Wall Crossing and Flat Connections

Pietro Longhi

Notes for lectures given at the Uppsala Geometry and Physics seminar, January 13 and 20, 2016.

Notice to the reader: these lecture notes are incomplete, deliberately approximative, and there are likely a few imprecisions lurking. The emphasis is on carving a bite-size line of reasoning through the literature on BPS spectroscopy, with a strong focus on the recent works of Gaiotto-Moore and Neitzke. Several important (in fact, crucial) other works are omitted, the reader is strongly encouraged to refer to the original sources for details and further references.

Version: January 26, 2016

※ ※ ※

Lecture 1

4d N=2 theories are an interesting subject, suitable for this seminar because of the deep connections they enjoy to several subjects in mathematics, such as: Donaldson-Thomas invariants, Motivic Hall algebras, quantization of Teichmueller spaces, integrable systems (in particular, Hitchin systems), cluster algebras, knot invariants and Chern-Simons theory.

1 BPS states in 4d N=2 theories

1.1 The algebra

The 4d $\mathcal{N} = 2$ super-Poincaré algebra contains even and odd subalgebras:

$$\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1 \tag{1}$$

respectively they are

$$\begin{aligned} &\mathfrak{s}_0 = \mathfrak{iso}(1,3) \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R \oplus \mathbb{C} \\ &\mathfrak{s}_1 = (1/2, 0; 1/2)_{+1} \oplus (0, 1/2; 1/2)_{-1} \qquad \text{as a } \underline{\text{rep}} \text{ of } \mathfrak{s}_0 \end{aligned} \tag{2}$$

A basis for $\mathfrak{s}^1$ is Q^A_α, $\overline{Q}_{\dot\beta B}$ (with $(Q^A_\alpha)^\dagger = \overline{Q}_{\dot\alpha A}$) which obey

$$\begin{aligned} \{Q^A_\alpha, \overline{Q}_{\dot\beta B}\} &= 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^B_A \\ \{Q^A_\alpha, Q^B_\beta\} &= 2\epsilon_{\alpha\beta}\epsilon^{AB}\overline{Z} \\ \{\overline{Q}_{\dot\alpha A}, \overline{Q}_{\dot\beta B}\} &= -2\epsilon_{\dot\alpha\dot\beta}\epsilon_{AB} Z \end{aligned} \tag{3}$$

the indices α, A etc. specify the transformation laws under other generators of the subalgebra $\mathfrak{s}_0$.

1.2 Representations

This is the symmetry algebra of the theory, physical states of the theory fall into representations of this algebra. We will be interested in particular in the Hilbert space of 1-particle states.

To describe massive particles, we use the method of induced representations. Going to the rest frame, we focus on the finite-dimensional representations of the "little" group of rotations $\mathfrak{so}(3) \subset \mathfrak{so}(1,3)$. Concretely, this means that we restrict to a vector sub-space of the full 1-particle Hilbert space, where $P^\mu|\Psi\rangle = M\delta^{\mu 0}|\Psi\rangle$.

In a suitable basis $\mathfrak{s}_1$ splits into two commuting Clifford algebras

$$\begin{aligned} \mathcal{R}^A_\alpha &= \xi^{-1} Q^A_\alpha + \xi \sigma^0_{\alpha\dot\beta} \overline{Q}^{\dot\beta A} \\ \mathcal{T}^A_\alpha &= \xi^{-1} Q^A_\alpha - \xi \sigma^0_{\alpha\dot\beta} \overline{Q}^{\dot\beta A} \end{aligned} \tag{4}$$

where $|\xi| = 1$, and

$$\{\mathcal{R}^A_\alpha, \mathcal{T}^B_\beta\} = 0\,. \tag{5}$$

Let $\mathfrak{h} = (j, j_R)$ be a representation of $so(3) \oplus su(2)_R$,[1] and let $|\Omega\rangle \in \mathfrak{h}$ be a state satisfying $\mathcal{R}^1_1|\Omega\rangle = \mathcal{R}^2_1|\Omega\rangle = 0$ (i.e. a "Clifford vacuum"), the remaining $\mathcal{R}$'s generate the Clifford multiplet[2]

$$|\Omega\rangle, \quad \mathcal{R}^1_2|\Omega\rangle, \quad \mathcal{R}^2_2|\Omega\rangle, \quad \mathcal{R}^1_2\mathcal{R}^2_2|\Omega\rangle \tag{6}$$

As a rep of the bosonic little group, this is a *half-hypermultiplet*

$$[(1/2,0) \oplus (0,1/2)] \equiv \rho_{hh} \tag{7}$$

Acting with $so(3) \oplus su(2)_R$ generators and with $\mathcal{R}$'s on $|\Omega\rangle$ we get a representation

$$\rho_{hh} \otimes \mathfrak{h} \tag{8}$$

Similarly, assuming $|\Omega\rangle$ is annihilated by half of the $\mathcal{T}$, we get another factor of ρ_{hh}.

$$\rho_{hh} \otimes \rho_{hh} \otimes \mathfrak{h} \tag{9}$$

1.3 Short representations

Some representations of this algebra have a special feature. Working out the rest of the anti-commutators gives

$$\begin{aligned} \{\mathcal{R}^A_\alpha, \mathcal{R}^B_\beta\} &= 4(M + Re(Z/\zeta))\epsilon_{\alpha\beta}\epsilon^{AB} \\ \{\mathcal{T}^A_\alpha, \mathcal{T}^B_\beta\} &= 4(-M + Re(Z/\zeta))\epsilon_{\alpha\beta}\epsilon^{AB} \end{aligned} \tag{10}$$

with $\zeta = \xi^{-2}$. Then choosing the basis for $\mathfrak{s}_1$ given by

$$\zeta = -e^{-i\alpha}\,, \text{for} \qquad \alpha = arg(Z) \tag{11}$$

[1] We will neglect $u(1)_R$ since it will be absent in physical theories of interest to us

[2] Here we use $\mathcal{R}^1_1 = (\mathcal{R}^2_2)^\dagger$, $\mathcal{R}^2_1 = (\mathcal{R}^1_2)^\dagger$

we get (from hermiticity property of $\mathcal{R}$'s) the (4d N=2) BPS bound

$$(\mathcal{R}_1^1 + (\mathcal{R}_1^1)^\dagger)^2 = 4(M - |Z|) \quad \Rightarrow \quad M \geq |Z| \tag{12}$$

States on which the bound is saturated are identically annihilated by the $\mathcal{R}$'s, they fall into *short* multiplets.

$$\begin{aligned} \text{long multiplets:} \quad & \rho_{hh} \otimes \rho_{hh} \otimes \mathfrak{h} \\ \text{short multiplets:} \quad & \rho_{hh} \otimes \mathfrak{h} \end{aligned} \tag{13}$$

2 4d N=2 gauge theories

The subject of 4d N=2 gauge theories is very rich, but not the main focus of these lectures, the discussion will be limited to a summary of the most relevant points. To get some intuition about generic features it suffices to focus on the simplest examples, we will choose pure SYM with gauge group SU(K).

2.1 UV description (N=2 SYM)

This theory has a unique UV lagrangian, parametrized by a complex coupling constant τ. The UV description involves a single superfield: an adjoint-valued N=2 vector-multiplet

$$(\varphi, \lambda, \psi, A_\mu) \tag{14}$$

The UV theory is described by the Lagrangian

$$\begin{aligned} \mathcal{L}_{UV} &= \frac{1}{8\pi} \mathrm{ImTr} \left[\tau \left(\int d^2\theta W^\alpha W_\alpha + 2 \int d^2\theta \, d^2\bar\theta \, \Phi^\dagger e^{-2V} \Phi \right) \right] \\ &= \frac{1}{g^2} \mathrm{Tr} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{g^2\,\theta}{32\pi^2} F_{\mu\nu} \tilde F^{\mu\nu} + (D_\mu\varphi)^\dagger D^\mu \varphi - \frac{1}{2} [\varphi^\dagger, \varphi]^2 \right. \\ &\quad \left. - i\lambda\sigma^\mu D_\mu \bar\lambda - i\bar\psi\bar\sigma^\mu D_\mu \psi - i\sqrt{2}[\lambda,\psi]\varphi^\dagger - i\sqrt{2}[\bar\lambda,\bar\psi]\varphi \right) \end{aligned} \tag{15}$$

(after integrating out the auxiliary fields). The bosonic part of this lagrangian is a Yang-Mills-Higgs theory, the vacuum equations for the scalar φ are

$$D_\mu \varphi = 0 \qquad [\varphi^\dagger, \varphi] = 0 \tag{16}$$

it is well known that this kind of model admits monopoles and dyons, and contains massive gauge bosons.

2.2 IR description (N=2 SYM)

At low energies, well below the scale of the lightest massive state, the effective action of the IR theory takes a different form. The most general lagrangian which is compatible with $\mathcal{N} = 2$ supersymmetry is

$$\mathcal{L}_{IR} = \frac{1}{8\pi} \mathrm{Im} \left(\int d^2\theta \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi^a \partial \Phi^b} W^{\alpha a} W^b_\alpha + \int d^2\theta d^2\bar\theta (\Phi^\dagger e^{2gV})^a \frac{\partial \mathcal{F}(\Phi)}{\partial \Phi^a} \right) , \tag{17}$$

the form of $\mathcal{F}$ is somewhat constrained by physical arguments, but its determination is nevertheless highly nontrivial. We will return to this point later.

2.3 Coulomb branch

A classical vacuum is characterized by a field configuration with φ valued in (some choice of) Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$, for example if $G = SU(2)$ we can take $\langle\varphi\rangle = diag(a, -a)$. More precisely, the gauge transformations in G/T change the choice of this sub-algebra, so they change the vacuum, but being gauge transformations their orbits are physically equivalent. On the other hand, different values of φ within $\mathfrak{t}$ are *inqeuivalent* vacua. However, there are elements of G/T which leave $\mathfrak{t}$ invariant, Weyl transformations, so the correct way to parametrize the vacuum moduli space is by "Casimirs" (Weyl-invariant polynomials) of φ. These are generated by the characteristic polynomial

$$\det(\lambda - \varphi) = \lambda^N - \frac{1}{2}\mathrm{Tr}\,(\varphi^2)\lambda^{N-2} - \frac{1}{3}\mathrm{Tr}\,(\varphi^3)\lambda^{N-3} + \dots \tag{18}$$

Classically the vacuum manifold is

$$\mathcal{B}_{class} = \mathfrak{t} \otimes \mathbb{C}/W \tag{19}$$

and it's parametrized by

$$u_i = \lim_{\vec{x}\to\infty} \mathrm{Tr}\,(\varphi(\vec{x})^i) \tag{20}$$

In a generic vacuum $u = (u_2, \dots, u_r)$, the IR theory is *abelian*, since the VEV $\langle\varphi\rangle$ breaks G to a maximal Cartan torus.

$$\mathcal{L}_{IR} \sim \int_{R^{1,3}} \mathrm{Im}(\tau_{IJ}) F^I \star F^J + \mathrm{Re}(\tau_{IJ}) F^I F^J + \dots \tag{21}$$

where ellipses are corrections encoded by $\mathcal{F}(u)$. Therefore we have a *family of theories*, of the Maxwell type controlled by a function of the moduli $\mathcal{F}(u)$. For this reason, the moduli space $\mathcal{B}$ is called the Coulomb branch, its dimension equals the rank of the gauge group.[3]

The low energy effective action $\mathcal{L}_{IR}$ induces a metric on $\mathcal{B}$[4]

$$ds^2 = \mathrm{Im}\,(\tau_{IJ}) da^I\, d\bar{a}^J \qquad \tau_{IJ} = \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J} \qquad \langle\varphi\rangle \sim diag(a^1 \dots a^r) \qquad u_k \sim \langle\varphi^k\rangle \text{ (for small } a). \tag{22}$$

The metric features some singularities, where the coupling constant of the theory blows up (the theory becomes strongly coupled). In the case of SU(2) SYM, this happens at $u = \pm\Lambda^2$. Physically, singularities are interpreted as massive states becoming massless. In fact in that case the low energy description is no longer valid (recall we assumed to be working below the scale of the lightest massive state).

[

2.4 Charge lattice

But for $u \in \mathcal{B}^* \equiv \mathcal{B} - \mathcal{B}^{sing}$ the IR theory is described by a *family of LEEAs* for the abelian degrees of freedom. For generic u the theory is, roughly speaking, r copies of supersymmetric Maxwell theory (with a lot of extra interactions). The abelian gauge symmetry $U(1)^r$ entails the existence

[3] Moduli spaces of vacua of theories with matter may also feature other branches, parametrized by VEVs of some of the matter fields.

[4] Substituting φ with its vev in the Lagrangian, we can think of it as gauged sigma model into $\mathcal{B}$.

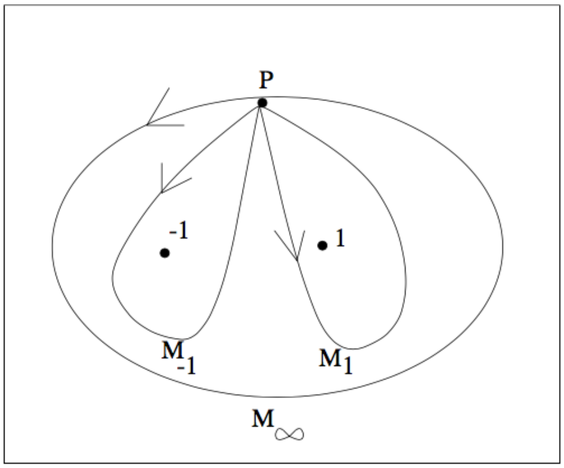

Fig. 1: The u plane with monodromies around 1, -1, and ∞. Note the choice of base point in the definition of the monodromies.

Figure 1:

of a corresponding global abelian symmetry which acts on the Hilbert space of the theory. Therefore the Hilbert space admits a grading by the electromagnetic charge: we introduce a *lattice*

$$\Gamma \simeq \mathbb{Z}^{\oplus 2rk(G)}. \tag{23}$$

then the Hilbert space of 1-particle states is graded accordingly

$$\mathscr{H} = \oplus_\gamma \mathscr{H}_\gamma . \tag{24}$$

2.5 Central charge as a topological quantity

The family of theories we are considering all enjoy 4d $\mathcal{N} = 2$ super-Poincaré symmetry. For each generator of the symmetry, there is a conserved current in the theory $\partial_\mu J^\mu = 0$. The conserved currents for the odd generators can be computed by the standard algorithmic Noether procedure

$$g^2 S^\mu_{(1)\,\alpha} = \sigma_{\nu\alpha\dot{\alpha}} \bar{\lambda}^{a\dot{\alpha}} (iF^{a\mu\nu} + \tilde{F}^{a\mu\nu}) + \sqrt{2}(\sigma^\nu \bar{\sigma}^\mu \psi^a)_\alpha D_\nu \varphi^{\dagger\, a} + \sigma^\mu_{\alpha\dot{\alpha}} \bar{\lambda}^{a\dot{\alpha}} (\varphi^\dagger T^a \varphi) \tag{25}$$

where the subscript (1) is for the R-symmetry index (the current is an $su(2)_R$ doublet), the second component $S^\mu_{(2)\,\alpha}$ is obtained by switching $\lambda \to \psi$ and $\psi \to -\lambda$. Now we can compute the algebra of conserved charges[5] as

$$
\begin{aligned}
\{Q_{(1)\alpha}, Q_{(2)\beta}\} &= \left\{ \int d^3x\, S^0_{(1)\alpha}(0,\vec{x}), \int d^3y\, S^0_{(2)\beta}(0,\vec{y}) \right\} \\
&= \dots \\
&= -\frac{2\sqrt{2}}{g^2}\epsilon_{\alpha\beta} \int d^3x\, \partial_i \left[(iF^{a\,0i} + \tilde{F}^{a\,0i})\varphi^{\dagger\,a} \right] \\
&= 2\sqrt{2}\epsilon_{\alpha\beta} \left(a n_e + \frac{4\pi}{g^2} a n_m \right)
\end{aligned}
\tag{26}
$$

where we used the fact that, at spatial infinity $F^a \varphi^{\dagger\,a}$ is the flux of the surviving abelian $U(1)^r$ photon, whose integral on S^2 at spatial infinity is the electric charge, while its dual gives the magnetic charge. Here a is the vev of φ at spatial infinity. This computation is done withouth taking into account the θ term, but its effect is to shift the electric charge (Witten effect) by

$$
n_e \to n_e + \frac{\theta}{2\pi} n_m \tag{27}
$$

giving the following expression for the central charge

$$
Z = a(n_e + \tau n_m) \qquad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}\,. \tag{28}
$$

One can repeat this reasoning in the low energy theory, whose lagrangian is controlled by $\mathcal{F}$, this gives

$$
Z = a n_e + a_D n_m \qquad a_D = \frac{\partial \mathcal{F}}{\partial a}\,. \tag{29}
$$

Note that since $\tau_{IR} = \partial^{\mathcal{F}}/\partial a$ we see that the expression approaches the "classical" one in the weak-coupling regime.

The take-away message from the above computation (due to Witten and Olive) is that in a 4d N=2 gauge theory, the central charge Z is realized as a topological charge. It depends linearly on the flux of the gauge field through the S^2 at spatial infinity: i.e. it depends linearly on $\gamma = (n_m, n_e)$

$$
Z \in Hom(\Gamma, \mathbb{C})\,. \tag{30}
$$

2.6 Charge and mass

In turn, any charged state carries a precise central charge Z_γ, and therefore its mass (the energy in the rest frame), which must obey $M \geq |Z_\gamma|$ is also related to the electromagnetic charge. BPS states saturate the bound. Furthermore, the central charge of a BPS state depends on $u \in \mathcal{B}$, since $a = a(u)$ and $a_D = a_D(u)$

$$
Z_\gamma(u)\,. \tag{31}
$$

This means that also its mass $M = |Z|$ depends on u. In fact varying u amounts to changing the 'coupling constants' of the IR theory, which are encoded in $\mathcal{F}$, therefore particles and their dynamics will have different properties, even different 'phases'.

[5]For non-physicists, make a remark that $\int^3_{\mathbb{R}} \partial_0 J^0 = \int^3_{\mathbb{R}} \partial_i J^i = \int_{S^2(\infty)} J^i = 0$

So for a BPS state, the charge γ determines much of the physical information, through Z_γ: the modulus gives the mass, and the phase gives the preserved supercharges (the $\mathcal{R}$).

Remark: BPS states become massless at $u = \pm\Lambda^2$, and are respondible for the singularities on $\mathcal{B}$.

2.7 BPS solitons in field theory

We started from a theory involving only a single superfield: the vector-multiplet. So what kind of *charged states* can appear in a pure SYM theory, without fundamental matter fields? *Supersymmetric solitons*, like BPS monopoles and dyons. A soliton is a finite-energy, static solution of the equations of motion. BPS solitons are field configurations that preserve part of the supersymmetry: if the central charge is $Z = e^{i\alpha}|Z|$ we expect to preserve $\mathcal{R}^A_\alpha$, this translates into the requirement that $[\mathcal{R}^A_\alpha, \psi] = [\mathcal{R}^A_\alpha, \lambda] = 0$, which gives the *BPS (soliton) equations*

$$\begin{aligned} F_{0\ell} - \frac{i}{2}\epsilon_{jk\ell}F_{jk} - iD_\ell(\varphi/\zeta) = 0 \\ D_0(\varphi/\zeta) - \frac{g}{2}[\varphi^\dagger, \varphi] = 0 \end{aligned} \tag{32}$$

where $\zeta = -e^{i\alpha}$.

These are the full BPS equations in the UV theory, but once a vev is picked the gauge symmetry is broken to an abelian one. Choosing an electromagnetic duality frame, the equations tell us that φ must be static (time-independent) and that

$$F^{+,I}_{0\ell} = i\partial_\ell(\varphi^I/\zeta) \tag{33}$$

where $I = 1\dots r$ is an electric index of the e.m. duality frame. To find solitonic, dyon-like solutions, let us make the ansatz in spatial polar coordinates

$$F^I = \frac{1}{2}(\rho^I_M \sin\theta\, d\theta \wedge d\phi + \rho^I_E \frac{dr \wedge dt}{r^2}) \tag{34}$$

where ρ^I_M, ρ^I_E are vectors in $\mathfrak{t}$ with components indexed by $I = 1, \dots, r$. Then plugging this ansatz into the fixed point equations, once can show that[6] the equations admit solutions if the VM moduli u carry space-dependence on r, as dictated by

$$2\mathrm{Im}\,[\zeta^{-1}\, Z_\gamma(u(r))] = \frac{\langle\gamma, \gamma_c\rangle}{r} + 2\mathrm{Im}\,[\zeta^{-1}\, Z_\gamma(u = \infty)] \qquad \forall\gamma \in \Gamma \tag{35}$$

where γ_c is the 'core' charge (as determined by the flux of the gauge field) and $\langle\cdot,\cdot\rangle$ is the DSZ pairing

$$\langle(n_m, n_e), (q_m, q_e)\rangle = n_m q_e - n_e q_m \tag{36}$$

(and its obvious generalization for higher rank gauge groups).

[6]see exercise 5.5 of PiTP lectures 2010 by Greg Moore

2.8 The Denef radius

Let's take a moment to contemplate this equation: the moduli depend on r, but in fact recall that the modulus u determines the mass of BPS states. Therefore, given a BPS state of charge γ_c, the mass of a second BPS state of charge γ_h will depend on its distance from γ_c! The energy of the system is then minimized for a special value of r, a computation gives (in the probe approximation when $M_{\gamma_c} \gg M_{\gamma_h}$)

$$E_{probe} = |Z_{\gamma_h}(u(r))|(1 - \cos(\alpha_h(r) - \alpha_c)) - \mathrm{Re}(Z_{\gamma_h}(u(r = \infty))/\zeta) \tag{37}$$

where $\alpha_h(r) = \arg(Z_{\gamma_h})(u(r))$ The energy is minimized for $\alpha_h(r) = \alpha_c$, therefore we find from (35)

$$r_{eq} = \frac{1}{2}\langle\gamma_h, \gamma_c\rangle \frac{1}{\mathrm{Im}\,(Z_{\gamma_h}\, e^{-i\alpha_c})} \tag{38}$$

where here Z_{γ_h} is understood to be evaluated at $r \to \infty$.

The physical interpretation is that, since masses are space-dependent, this is equivalent to an effective potential energy. The gradient of the energy results in an effective attractive/repulsive force, which is null at r_{eq}.

A crucial feature of this boundstate is that it has been shown to be BPS, so *two BPS particles can form a BPS boundstate.*

Beyond the probe approximation, the case of two particles has been solved by Denef, his boundstate radius formula is

$$r_{Denef}(u) = \frac{1}{2}\langle\gamma_1, \gamma_2\rangle \frac{|Z_{\gamma_1}(u) + Z_{\gamma_2}(u)|}{\mathrm{Im}\,(Z_{\gamma_1} Z^*_{\gamma_2}(u))} \tag{39}$$

where α_{1+2} is the phase of $Z_{\gamma_1+\gamma_2}$. Quite important is that the boundstate is itself a BPS state!

3 Wall-crossing

In the deep IR the theory is one of (supersymmetric) abelian "photons", which don't interact with each other. Dynamics gets more interesting once the lightest interacting states are taken into account: these are the charged BPS solitons. So to understand the IR dynamics it's important to know *which* BPS solitons states appear in the Hilbert space: what are their charges γ and their representations (j, j_R)? Physicists call this the problem of *computing the BPS spectrum.* Before we turn to how the BPS spectrum is computed, we must sharpen the question by discussing an important property of BPS spectra of 4d N=2 gauge theories.

3.1 Marginal Stability

BPS states are charged particles, and we have seen that they interact with each other. A word of warning: not quite a Coulomb type force, we are in a supersymmetric setting, which involves several carriers of the 'force' (vector bosons, scalars, gauginos). In fact they can bind together, into a boundstate which is itself BPS, the binding energy of a bounstate is then

$$\Delta E = M_\gamma + M_{\gamma'} - M_{\gamma+\gamma'} = |Z_\gamma| + |Z_{\gamma'}| - |Z_{\gamma+\gamma'}| \geq 0\,, \tag{40}$$

because $Z_{\gamma+\gamma'} = Z_\gamma + Z_{\gamma'}$.

As we have seen, the mass M depends on $u \in \mathcal{B}^*$, since it's generated dynamically. As u varies, the binding energy can go to zero, for such choices of vacua the boundstates become *marginally stable* and can decay or form. Because of the BPS condition, this happens when

$$\arg(Z_\gamma) = \arg(Z_{\gamma'})\,. \tag{41}$$

from the Denef radius formula, we see in fact that $r_{eq} \to \infty$ in this limit, the boundstate splits up! The radius formula tells us more: for certain choice of u, the radius become negative, meaning that the boundstate becomes unstable.

The marginal stability condition defines real-codimension 1 loci called *walls of marginal stability* $MS(\gamma, \gamma')$

$$MS(\gamma, \gamma') = \{u \in \mathcal{B}\,,\ \arg(Z_\gamma(u)) = \arg(Z_{\gamma'}(u))\} \tag{42}$$

MS walls divide $\mathcal{B}$ into chambers, and the BPS spectrum differs from a chamber to the other, since BPS bounstates appear or disappear across these walls. The BPS spectrum is thus piecewise constant on $\mathcal{B}$. This the wall-crossing phenomenon, and it plays an important role in answering the question "what's the BPS spectrum?".

3.2 The Wall Crossing Formula

The change in the BPS spectrum is described by a rather curious formula, discovered in 2008 by Kontsevich and Soibelman. The appearance of this formula sparked much progress, since then the computation of BPS spectra took off, and many examples have been computed as of today.

The basic ingredients of the formula are:

- A lattice Γ or rank $2r$, equipped with $\mathbb{Z}$-valued symplectic pairing $\langle\cdot,\cdot\rangle$
- A central charge function $Z \in Hom(\Gamma, \mathbb{C})$
- A "quadratic refinement" $\sigma : \Gamma \to \mathbb{Z}_2$ such that $\sigma(\gamma_1)\sigma(\gamma_2) = (-1)^{\langle\gamma_1,\gamma_2\rangle}\sigma(\gamma_1+\gamma_2)$
- The BPS index $\Omega(\gamma, u) := -\frac{1}{2}\mathrm{Tr}_{\mathscr{H}_\gamma^{BPS}}(-1)^{2J_3}(2J_3) \equiv \mathrm{Tr}_{\mathfrak{h}}(-1)^{2J_3}$

We consider a complexified torus $T_u = \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*)^{2r}$. For each $\gamma \in \Gamma$ there is a function X_γ on T_u. Upon choosing a basis $\gamma^1 \ldots \gamma^{2r}$ for Γ, we can choose coordinates $X^i = X_{\gamma_i}$ for T_u. The X_γ obey

$$X_\gamma X_{\gamma'} = X_{\gamma+\gamma'}. \tag{43}$$

The symplectic DSZ pairing $\langle\cdot,\cdot\rangle$ on Γ^* gives a homolorphic symplectic form on T_u

$$\omega_T = \frac{1}{2}\langle\gamma^i, \gamma^j\rangle\,\frac{dX^i}{X^i} \wedge \frac{dX^j}{X^j} \tag{44}$$

The quadratic refinement can be chosen at will, one possible choice is $\sigma(\gamma) = (-1)^{\langle\gamma_e,\gamma_m\rangle}$ where $\gamma = \gamma_e + \gamma_m$ according to a choice of e.m. splitting for Γ (valid only locally on $\mathcal{B}$).

Then define the transformation

$$\mathcal{K}_\gamma : X_{\gamma'} \mapsto X_{\gamma'}(1 - \sigma(\gamma)X_\gamma)^{\langle\gamma',\gamma\rangle} \tag{45}$$

which preserves the symplectic structure of T_u.

Now choose any $u \in \mathcal{B}$, at such point we have a central charge function $Z \in Hom(\Gamma, \mathbb{C})$. We construct the following phase-ordered product

$$\mathbb{S}_u =: \prod_{\gamma \in \Gamma} \mathcal{K}_\gamma^{\Omega(\gamma,u)} : \tag{46}$$

where the : : denote that the product is ordered according to the phase of Z_γ. As u is in some chamber of $\mathcal{B}$, the phase-ordering is constant. When u crosses a wall of marginal stability, the phase-ordering jumps. The statement of the WCF is that the BPS indices (hence the BPS Hilbert space) also jump, and do so in precisely a way such that

$$\mathbb{S}_u = \mathbb{S}_{u'} \tag{47}$$

this product is *constant*.

3.3 Examples

Example 1 Argyres-Douglas theory: here Γ is rank two, and there is a chamber with two BPS states of charges γ_1, γ_2 with $\langle \gamma_1, \gamma_2 \rangle = 1$ and $\Omega = 1$ for both. Then

$$\mathbb{S}_u = \mathcal{K}_{\gamma_1} \mathcal{K}_{\gamma_2} . \tag{48}$$

On the other side of the MS wall, the phases of Z_{γ_1} and Z_{γ_2} swap, the KSWCF preditcs

$$\mathbb{S}_{u'} = \mathcal{K}_{\gamma_2} \mathcal{K}_{\gamma_1+\gamma_2} \mathcal{K}_{\gamma_1} . \tag{49}$$

One boundstate appears, with $\Omega = 1$.

Example 2 Seiberg-Witten SU(2) SYM theory: here Γ is rank two, and there is a chamber with two BPS states of charges γ_1, γ_2 with $\langle \gamma_1, \gamma_2 \rangle = 2$ and $\Omega = 1$ for both. Then

$$\mathbb{S}_u = \mathcal{K}_{\gamma_1} \mathcal{K}_{\gamma_2} . \tag{50}$$

On the other side of the MS wall, the phases of Z_{γ_1} and Z_{γ_2} swap, the KSWCF preditcs

$$\mathbb{S}_{u'} = \mathcal{K}_{\gamma_2} \ldots \mathcal{K}_{\gamma_2+k(\gamma_1+\gamma_2)} \ldots \mathcal{K}_{\gamma_1+\gamma_2}^{-2} \ldots \mathcal{K}_{\gamma_1+k(\gamma_1+\gamma_2)} \ldots \mathcal{K}_{\gamma_1} . \tag{51}$$

Infinitely many boundstates appear.

Exercise: Prove that $\mathcal{K}_\gamma \mathcal{K}_{\gamma'} = \mathcal{K}_{\gamma'} \mathcal{K}_\gamma$ if the pairing is zero. Also, what happens if the pairing is 3, or higher?

A nice way of visualizing the wall-crossing phenomenon is the following. Associate to each BPS particle in the spectrum at u a ray in the ζ complex plane

$$\ell_\gamma := \{\zeta : Z_\gamma(u)/\zeta \in \mathbb{R}_-\} \tag{52}$$

As long as u is not on a MS wall, all rays are distinct.

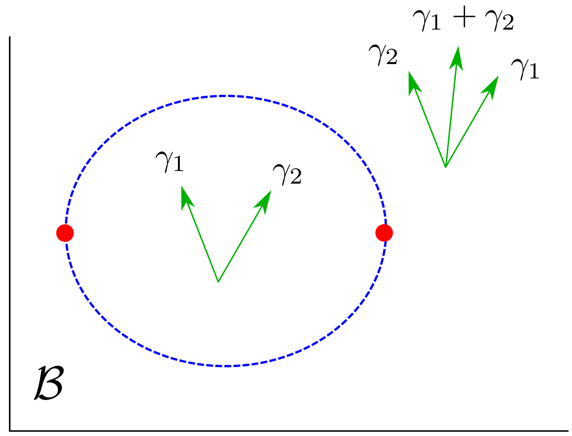

Figure 2: The u plane in Argyres Douglas theory. Walls of marginal stability in blue. 2 BPS states inside, 3 outside (+ CPT conjugates).

3.4 How KS appears in physics

A short overview of what we will talk about next week.

- Where are the complex tori in physics? The Coulomb branch $\mathcal{B}$ is the base of a torus fibration $\mathcal{M} \to \mathcal{B}$, the X_γ can be thought as coordinates on $\mathcal{M}$.
- How does the KSWCF arise in physics? The metric on $\mathcal{M}$ should be smooth across MS walls. Smoothness implies the KSWCF.
- OK we know how the BPS spectrum *jumps*, but still we need the BPS spectrum *somewhere* on $\mathcal{B}$, to make use of the KSWCF. I will introduce spectral networks, and briefly comment on how they compute BPS spectra and interplay with the KSWCF.

References

Greg Moore, PiTP 2010 lectures on Wall crossing, available at his webpage
Alvarez-Gaumé and Hassan, lecture notes on Seiberg-Witten theory, can be found on arxiv.
Gaiotto, Moore, Neitzke, "Four-dimensional wall crossing via three-dimensional field theory"

Lecture 2

4 Physical origin of the KSWCF: 3d $\mathcal{N} = 4$ σ-models

One strategy to study the BPS spectrum is to consider a 4d N=2 theory on $\mathbb{R}^3 \times S^1$ on a time-like circle of finite radius R. Compactification on a circle preserves all supercharges, so the algebraic considerations on BPS multiplets made at the beginning still hold. At low energies the theory is described by a 3d N=4 sigma model, with a hyperkahler target space $(\mathcal{M}, g)$. The sigma model receives corrections from BPS instantons, in which the worldline of a BPS particle of the 4d theory wraps the time-like S^1. At large radius R, the corrections to the path integral will be of the form $e^{-2\pi R|Z_\gamma|}$ for single-particle BPS states. The key idea here: if we knew the metric g, we could extract the BPS spectrum from its R-dependence, at least in principle.

4.1 3d low energy effective action at large radius

Again, we stick to N=2 SYM for simplicity. If Λ is the dynamical IR scale, we study the theory on a circle, at energies $\mu \ll 1/R \ll \Lambda$. For large radius, the dynamical 3d degrees of freedom are the x^0-independent modes of the 4d fields. The bosonic degrees of freedom are thus

$$\begin{aligned} \text{scalars} \quad & a^I(\vec{x}) \\ \text{electric Wilson lines (holonomies)} \quad & \theta_e^I(\vec{x}) = \oint A_0(\vec{x})dx^0 \\ \text{magnetic ones} \quad & \theta_m^I(\vec{x}) = \oint A_{D,0}(\vec{x})dx^0 \end{aligned} \tag{53}$$

where the dual gauge field $A_{D,0}$ is really a scalar on $\mathbb{R}^3$, its divergence is Hodge-dual to the fieldstrength of $A_i dx^i$. The θ are *periodic scalars* and coordinatize a $2r$-torus $\mathcal{M}_u \sim (\mathbb{R}/2\pi\mathbb{Z})^{2r}$.

The 3d theory is then a sigma model into a target space $\mathcal{M}$, of real-dimension $4r$, which is topologically a $2r$-torus fibration over $\mathcal{B}$. The theory has $N = 4$ supersymmetry (8 supercharges in 3d), which implies that the metric on $\mathcal{M}$ is hyperkahler.[7]

The effective 3d action is, in terms of the 3d degrees of freedom:

$$\begin{aligned} \mathcal{L}_{IR}^{3d} &= (\operatorname{Im}\tau)\left(-\frac{R}{2}|da|^2 - \frac{R}{2}F^{(3)} \wedge \star F^{(3)} - \frac{1}{8\pi R}d\theta_e^2\right) + (\operatorname{Re}\tau)\left(\frac{1}{2\pi}d\theta_e \wedge F^{(3)}\right) \\ &\quad + (\text{Fermi}) \\ &= -\frac{R}{2}(\operatorname{Im}\tau)|da|^2 - \frac{1}{8\pi^2 R}(\operatorname{Im}\tau)^{-1}|dz|^2 + (\text{Fermi}) \end{aligned} \tag{54}$$

where[8]

$$dz_I = d\theta_{m,I} - \tau_{IJ} d\theta_e^J \,. \tag{55}$$

[7] For example, Seiberg and Witten showed that taking $R \to 0$ with $G = SU(2)$, then $\mathcal{M}$ is the Atiyah-Hitchin manifold. But we are interested in the opposite limit to compute contributions of BPS states.

[8] To be precise, dz_I is not a closed form globally on $\mathcal{M}$, but only on the fiber $\mathcal{M}_u$.

This is in fact a supersymmetric sigma model, with target space metric

$$g_{sf} = R\,(\mathrm{Im}\,\tau)|da|^2 + \frac{1}{4\pi^2 R}(\mathrm{Im}\,\tau)^{-1}|dz|^2 \tag{56}$$

This is called the "semiflat metric" because fibers are flat tori, also this expression makes it manifest that g_{sf} is Kahler with respect to the complex structure where da_I, dz_I are a basis for holomorphic 1-forms $\Omega^{1,0}(\mathcal{M})$. In this complex structure, the fibers are complex tori, and this is called the "Seiberg-Witten fibration".

Remark for physicists: the fibers are, roughly speaking, the Jacobian of the Seiberg-Witten curve. For Mathematicians: we will discuss later what is the Seiberg-Witten curve.

The fact that $\mathcal{M}$ is a torus fibration is reminiscent of the structure of an integrable system, we will return to this point below.

4.2 Quantum corrections from BPS states

The semiflat metric is derived purely from the 4d LEEA, by taking constant modes along the circle. But, it is singular over loci in the base $\mathcal{B}^{sing}$.

One approach to constructing a "complete" metric on $\mathcal{M}$ is Hitchin's twistor construction: one considers the twistor space

$$\mathcal{Z} = \mathcal{M} \times \mathbb{CP}^1 \to \mathbb{CP}^1 \tag{57}$$

where $\mathbb{CP}^1$ is the sphere of complex structures J_ζ, with coordinate ζ, and the fiber above each ζ is $\mathcal{M}$ in the corresponding complex structure. Hitchin's theorem: a hyperkahler metric g is equivalent to a *holomorphic symplectic form* on $\mathcal{M}$

$$\varpi = \frac{1}{\zeta}\omega_+ + \omega_3 + \zeta\omega_- \,. \tag{58}$$

That is, given such ϖ we can read off the 3 two-forms which define the hyperkahler structure.

To make use of this, we make an assumption: $\mathcal{M}$ has a coordinate atlas, with *local charts* of the form

$$\mathcal{U} \simeq \Gamma^* \otimes \mathbb{C}^* \simeq (\mathbb{C}^*)^{2r} \tag{59}$$

with local "Darboux coordinates" $X_\gamma \in \mathbb{C}^*$ satisfying

$$X_\gamma X_{\gamma'} = X_{\gamma+\gamma'} \,. \tag{60}$$

as well as a list of technical properties (...see original reference: Gaiotto Moore Neitzke "Four dimensional wall-crossing via three dimensional field theory").

If these Darboux functions exist, then we can write a holomorphic-symplectic form in the following way. Choosing a basis γ^i of Γ

$$\varpi(\zeta) = \frac{1}{8\pi^2 R}\langle\gamma_i, \gamma_j\rangle \frac{dX_{\gamma_i}}{X_{\gamma_i}} \wedge \frac{dX_{\gamma_j}}{X_{\gamma_j}} \tag{61}$$

the properties of the X_γ guarantee that this is globally defined, and holomorphic in ζ. The key point here is: to build the metric through the Twistor approach, we can actually shift our focus on finding Darboux functions $X_\gamma(u, \theta; \zeta)$ subject to a certain set of properties. The coordinates then give the holomorphic symplectic form, and from it we can extract the HK structure.

Side remark: these local charts are precisely the kind of torus that appears in the KSWCF setting, with the symplectic structure also the same as we saw above.

For example, the semiflat metric can be constructed in this way. The appropriate Darboux coordinates are

$$X^{sf}_\gamma = \exp\left[\pi R \zeta^{-1} Z_\gamma + i\theta_\gamma + \pi R \zeta \bar{Z}_\gamma\right] \tag{62}$$

In this case one finds

$$\omega^{sf}_+ = \frac{1}{2\pi} da^I \wedge dz_I \tag{63}$$

hence J_3 is the complex structure that we used above in the Seiberg-Witten fibration. Moreover

$$\omega^{sf}_3 = \frac{i}{2}\left(R(\mathrm{Im}\tau)_{IJ} da^I \wedge d\bar{a}^J + \frac{1}{4\pi^2 R}((\mathrm{Im}\tau)^{-1})^{IJ} dz_I \wedge d\bar{z}_J\right) \tag{64}$$

this clearly gives back the semiflat metric.

Taking into account quantum corrections, the story is more complicated. In at least one case ($U(1)$ theory coupled to a fundamental hypermultiplet) it is possible to compute the metric, and hence the X_γ explicitly by field theoretic techniques.

More generally, it is possible to characterize the X_γ by a set of differential equations (justifying the appearance of a Stokes phenomenon, see later) they must obey, the result is a 4d version of the tt^* equations on the ζ plane. (differential equations for the manifold of vacua of a 4d theory). The solution is implicitly given by the following integral equations

$$X_\gamma(\zeta) = X^{sf}_\gamma(\zeta) \exp\left[-\frac{1}{4\pi i}\sum_{\gamma'} \Omega(\gamma', u)\langle\gamma,\gamma'\rangle \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta} \log\left(1-\sigma(\gamma')X_{\gamma'}(\zeta')\right)\right] \tag{65}$$

where

$$\ell_\gamma = \{\zeta | Z_\gamma/\zeta < 0\}\,. \tag{66}$$

The coordinates explicitly depend on the BPS spectrum, in fact we expect that the metric will too. If we know the metric, we get the spectrum, and vice versa.

Notice a property of the $X_\gamma(\zeta)$: they are piecewise holomorphic in ζ, because as ζ crosses one of the ℓ_γ, we have

$$X_{\gamma_0} \to \mathcal{K}^{\Omega(\gamma)}_\gamma X_{\gamma_0} \tag{67}$$

the jump is by a KS transformation.[9]

Also notice that $\Omega(\gamma, u)$ which enters the equation for the X depend on $u \in \mathcal{B}$. So one may worry that the metric g on $\mathcal{M}$ may jump at MS walls on the base $\mathcal{B}$ – but the KSWCF guarantees that the metric is smooth.

[9]Stokes phenomenon on the ζ plane, the diff eqs are the "4d tt^* equations" which I didn't write

5 Class $\mathcal{S}$ theories and Hitchin systems

5.1 Class S data

So far we talked about SYM, now we enlarge our focus to a larger set of theories, known as class $\mathcal{S}$. A class $\mathcal{S}$ theory is a 4d N=2 theory specified by a triplet of data $(\mathfrak{g}, C, D)$: a simple and simply laced Lie algebra $\mathfrak{g}$ (in the case of SYM it's the algebra of the gauge group, but in general it's not), a punctured Riemann surface C, and certain "data" D attached to the punctures.

Given this data, one can construct a 4d N=2 theory by twisted compactification of the 6d (2,0) theory on C, with boundary conditions specified by D for worldvolume fields at the punctures. Hence the name "S", for "Six". (The 6d theory is actually not yet constructed, not even by physical standards, but many of its properties can be inferred from its relation to string/M-theory.) These theories have been very popular since their discovery (2009), and much progress has been made on them in the past few years.

5.2 Hitchin systems

This progress includes wall-crossing, and BPS spectroscopy. In this regard, the starting point is to note that the same triplet of data can be employed to formulate the Hitchin equations on C (remark to physicists: these are the self-duality equations reduced on a Riemann surface, with a twist, as originally derived by Hitchin himself)

$$F + R^2[\varphi, \overline{\varphi}] = 0 \qquad \bar{\partial}\varphi + [A_{\bar{z}}, \varphi] = 0 \tag{68}$$

where A is a $\mathfrak{g}$-valued connection on C and φ is a section of $(K_C \otimes \mathfrak{t})/W$.

The moduli space of solutions to these equations is the Hitchin moduli space $\mathcal{M}_H$: a hyperkahler manifold, with the additional structure of an algebraic integrable system.[10] Concretely, there is a lagrangian fibration $\mathcal{M}_H \to \mathcal{B}_H$. The 6d origin of class S theories identifies $\mathcal{B}_H \equiv \mathcal{B}$ and $\mathcal{M}_H \equiv \mathcal{M}$. In this setting, the base $\mathcal{B}$ is parametrized by Casimirs of $\varphi(z)$, the k-differentials

$$\{\phi_k(z) = \mathrm{Tr}\,(\varphi(z)^k)\}_k \quad \leftrightarrow \quad \{u_i\}_i\,. \tag{69}$$

i.e. $\mathcal{B} \simeq \bigoplus_k H^0(C, K^{\otimes k})$

Given a solution to the equations, we can construct the spectral curve

$$\Sigma_\rho(u)\ :\ \det{}_\rho(\lambda - \varphi(z)) = 0 \subset T^*C \tag{70}$$

where λ is the canonical 1-form on T^*C. The spectral curve is usually taken in the 1st fundamental representation; it only depends on $u \in \mathcal{B}$ and not on the fiber coordinates of $\mathcal{M}$. So the Hitchin equations really define a family of Riemann surfaces fibered over $\mathcal{B}$. The embedding into T^*C implies a natural presentation of Σ

$$\pi : \Sigma \xrightarrow{d:1} C \tag{71}$$

as a ramified covering of C.

[10] Complete integrability means that there is a maximal (half-dimensional) set of functionally independent and Poisson-commuting hamiltonians. "Algebraic" means that the fiber above each point of the Hamiltonian base is (roughly) isomorphic to an abelian variety.

5.3 Seiberg-Witten description

The spectral curve Σ is what physicists call the "Seiberg-Witten curve" of the 4d N=2 theory. Seiberg and Witten discovered that certain theories (4d N=2) admit a nice geometric description in the IR, which captures many nontrivial aspects of the physics (like dualities). The Seiberg-Witten curve contains much information about the IR dynamics of the 4d theory, in particular

- The lattice of electromagnetic charge is identified with the homology of Σ

$$\Gamma \simeq H_1(\Sigma, \mathbb{Z}) \tag{72}$$

  Then electromagnetic charges γ are cycles on Σ, and

$$\text{DSZ pairing} \quad \leftrightarrow \quad \text{intersection pairing} \tag{73}$$

- The central charge function $Z \in Hom(\Gamma, \mathbb{C})$ is given by

$$Z_\gamma(u) = \frac{1}{\pi} \oint_\gamma \lambda(u) \tag{74}$$

- The family of LEEAs, which is determined by $\mathcal{F}(u)$, is completely fixed by the fibration (Σ_u, λ_u) over $\mathcal{B}$.

- At singularities on $\mathcal{B}$, massless states appear – this means that some $Z_\gamma \to 0$ and in fact the cycle γ "pinches" at u

Example In SU(2) SYM the curve C is a cylinder, with irregular singularities at $z = 0, \infty$. $\varphi(z)$ is a 2×2 matrix, and $\Sigma \to C$ is a 2-fold ramified covering

$$\lambda^2 - \phi_2(u; z) = \lambda^2 - \left(\frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z} \right) (dz)^2 = 0 \tag{75}$$

there are two square-root branch points, and Σ is topologically a 2-torus. The homology lattice is rank 2, generated by γ_e, γ_m, and this is in line with the expectations since the IR theory is a U(1) theory. The fibration degenerates when $u = \pm\Lambda^2$.

The Coulomb branch is the one we already saw: the complex plane with 2 singularities at $u = \pm\Lambda^2$.

Example In ADE-type SYM the curve C is always a cylinder, with irregular singularities at $z = 0, \infty$. The cover $\Sigma \to C$ is $d : 1$ where d is the dimension of the first fundamental representation. General expressions are explicitly known and can be readily found in the literature. One finds that there are $2r$ square-root-type branch points, plus higher-type ramification at $z = 0, \infty$. A simple counting reveals that the rank of H_1 exceeds that of Γ, then one is face with the task of identifying the physical definition of Γ as a sub-quotient of $H_1(\Sigma)$.

The general answer to this problem comes from an interesting construction of Ron Donagi, known as *Cameral cover*, but we will not go into that in these lectures. It would be interesting to understand the physical role of cameral covers better.

5.4 Flat connections

The Hitchin moduli space $\mathcal{M}$ is *also* a moduli space of flat connections because, if (φ, A) is a solution of the equations, then

$$\mathcal{A} = \frac{R}{\zeta}\varphi + A + R\zeta\bar{\varphi} \tag{76}$$

is a flat connection

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0 \tag{77}$$

so

$$\mathcal{M}_H \simeq \mathcal{M}_F(\mathfrak{g}_{\mathbb{C}}, C)\,. \tag{78}$$

Moreover, it turns out that points in the fiber $\mathcal{M}_u$ parametrize flat line bundles $\mathcal{L} \to \Sigma$, so $\mathcal{M}$ can also be thought (roughly) as a moduli space of flat abelian connections on Σ, at least locally

$$\mathcal{M}_H \simeq \mathcal{M}_F(GL(1,\mathbb{C}), \Sigma)\,. \tag{79}$$

5.5 Two coordinate systems

Physical reasoning motivates to consider two different sets of coordinates on $\mathcal{M}$ (really, functions on the twistor space of $\mathcal{M}$):

$$F(\wp, \zeta), X_\gamma(\zeta)\,. \tag{80}$$

These arise in physics as vevs of UV and IR supersymmetric line defects.

- A UV supersymmetric line defect is a BPS "Wilson line" (or a generalization thereof), in class S these have been *partially classified*.
  - But some of them are parametrized by closed paths $\wp \subset C$.
  - The defect preserves supercharges $\mathcal{R}^A_\alpha(\zeta)$, so its vev will be a function on the moduli space of vacua $\mathcal{M}$ that will be *holomorphic* in complex structure $J(\zeta)$.
  - Schematically

    $$F(\wp, \zeta) = \langle L_{UV}(\wp, \zeta)\rangle \sim \mathrm{Tr}\,_\rho P \exp \oint_\wp \nabla^{n.a.} \tag{81}$$

    so they are coordinates on $\mathcal{M}_F(\mathfrak{g}_{\mathbb{C}}, C)$ as they are holonomies of a non-abelian connection $\mathcal{A}$ of the Hithcin system on C.

- Similarly for IR line defects
  - The IR defects are labeled by e.m. charges $\gamma \in \Gamma$ since the gauge group is Abelian. By the Seiberg-Witten picture these are homology cycles on Σ.
  - An IR supersymmetric line defect is a BPS Wilson line with e.m. charge γ, and preserving supercharges $\mathcal{R}^A_\alpha(\zeta)$. Its vev will be a function on the moduli space of vacua $\mathcal{M}$ that will be *piecewise* holomorphic in complex structure $J(\zeta)$.
  - In fact the $X_\gamma(\zeta)$ exhibit jumps in correspondence of K-walls

    $$\{(u, \zeta); Z_\gamma(u)/\zeta < 0\} \tag{82}$$

    (recall the TBA equation for X_γ)

– Schematically

$$X_\gamma = \langle L_{IR}(\gamma, \zeta)\rangle \sim P \exp \oint_\gamma \nabla^{ab} \tag{83}$$

they are coordinates for $\mathcal{M}_F(GL(1,\mathbb{C}), \Sigma)$, they can be thought as holonomies ∇^{ab}.

This interpretation comes from the M theory picture, in which class S theories arise as world-volume low energy descriptions of stacks of M5 branes (valid for A-type only). In that context, line defects arise from M2 branes ending on M5 along $S^1 \times \wp \subset S^1 \times \mathbb{R}^3 \times C$. They couple to the 2-form on the M5, which descends to (non)abelian connections in 4d and on the Riemann surface $(\Sigma)C$. Part of this story is conjectural, but the above properties can be inferred from the M theory picture. We will not go into the details for lack of time.

5.6 Interpretation via framed wall-crossing

Since $X_\gamma, F(\wp)$ are really coordinate systems on the same manifold, they are both holomorphic in J_ζ, they must be related somehow. There is a physical interpretation of this relation, in terms of "Framed BPS states": roughly

$$F(\wp, \zeta) = \sum_\gamma \overline{\underline{\Omega}}(\wp, \gamma, u, \zeta) X_\gamma \tag{84}$$

where the $\overline{\underline{\Omega}}$ are integers called "framed BPS degeneracies".

A motivation for this kind of formula is roughly the following

- Can write $\langle L_{UV}(\zeta)\rangle = \mathrm{Tr}_{\mathscr{H}_{L(\zeta)}} (-1)^F e^{-2\pi R H} e^{i\theta\cdot Q}\sigma(Q)$ as a trace over the Hilbert space.
- In the $R \to \infty$ limit, this trace is projected on the lowest energy states: framed BPS states. Moreover they come in the form

$$\langle L_{UV}(\zeta)\rangle \sim \sum_\gamma \overline{\underline{\Omega}}(L(\zeta); u)\, e^{2\pi R\, \mathrm{Re}(Z_\gamma/\zeta) + i\theta_\gamma} = \sum_\gamma \overline{\underline{\Omega}}(L(\zeta); u)\, Y_\gamma \tag{85}$$

  where Y_γ have the right asymptotics at large R to resemble the X_γ.
- The wall-crossing of $L(\zeta)$ should be trivial, because there is no phase transition in the UV theory as we vary u.
- On the other hand, the wall-crossing of the $\overline{\underline{\Omega}}$ can be computed by quantizing halos of 4d BPS particles orbiting around the defect: halos can come and go (recall the radius formula from previous lecture), this is the framed wall-crossing.
- One can show that the jump in the $\overline{\underline{\Omega}}$ leaves $\langle L_{UV}(\zeta)\rangle$ invariant if the Y_γ jump at K-walls precisely as the X_γ do (see the behavior discussed above, after the integral equations for X_γ)

Example: SU(2) SYM Wilson line has vev (will explain later how to compute this via spectral networks)

$$F(\wp) = X_{\gamma_e} + \frac{1}{X_{\gamma_e}} + X_{\gamma_m+\gamma_e} \tag{86}$$

the first two terms, when expanded in the $R \to \infty$ limit, are the expected ones semicalssically. The last term is a nontrivial nonperturbative correction, this VEV can be checked by field theoretic considerations (Daniel Brennan has been working on this).

As the vacuum u of the IR theory is changed across a K-wall, the framed BPS spectrum jumps. There will be new "halos" (i.e. $so(3)$ multiplets) of framed BPS states appearing or disappearing, so the $\overline{\underline{\Omega}}$ jump. The X_γ also jump, by a KS transformation (recall discussion below the integral equations for X_γ's.). The result is that the LHS is constant – and this makes good sense physically, because the UV physics should be independent of IR moduli.

Key messages from the picture of framed wall-crossing:

- Line defects serve as "probes" of the 4d BPS states. In particular, the relation of $F(\wp)$ to the X_γ gives the $\overline{\underline{\Omega}}$, and from the wall-crossing properties of these it is possible to extract the "vanilla" BPS degeneracies Ω. (By studying jumps at K-walls, but we will return to this in detail with spectral networks).

- Vevs of line defects provide two different coordinate systems of $\mathcal{M}$, each can be viewed as holonomies of flat (non)abelian connections on $(\Sigma)\, C$. BPS states are, in some way, the key ingredients in the relation between the two coordinate sets. Will make this precise right next, with spectral networks.

The point on where we stand, regarding computing the BPS spectrum
At the beginning of this lecture we emphasized that BPS states contribute quantum corrections to $(\mathcal{M}, g)$, so BPS indices can be "extracted" from g. But this was not an effective way to get BPS spectra, since g is hard to compute. In fact, this relation is most useful if applied the other way around (computing g from knowledge of the spectrum).
Now we have changed strategy, we study two coordinate systems on $\mathcal{M}$: one of them is the "Darboux coordinates" X_γ, the other is the coordinate set of non-abelian holonomies $F(\wp, \zeta)$. Their relation encodes the BPS spectrum, via framed wall-crossing. Next we will see precisely how.

6 Spectral Networks - the fun part

Consider the following question: given a flat line bundle on Σ, how do we push it forward to C? At generic points, we can just take the fiber $E_z \simeq \oplus_\nu \mathcal{L}_{\nu,z}$. But there is an obvious obstruction at the branch points.

Spectral networks provide a systematic answer to this. In so doing, they also compute BPS indices, for the reasons we outlined in previous sections.

6.1 Generalities on spectral covers

Consider a ramified covering $\pi : \Sigma \to C$, with Σ arising as above from the spectral equation of a Hitchin system.

$$\det_\rho(\lambda - \varphi(z)) = 0 \tag{87}$$

We will take $\mathfrak{g} = ADE$ and ρ to be minuscule. All fundamental reps for A_n, the vector and the spinors for D_n; the $27, \overline{27}$ for E_6; and the 56 for E_7. None for E_8.

- Above z, the fiber is $\pi^{-1}(z) \simeq \Lambda_\rho$. Sheets are 1-1 with weights of ρ,

$$\lambda_i(z) = \langle \nu_i, \varphi(z) \rangle \in T_z^* C \tag{88}$$

- The sheet monodromy around a generic closed path $\wp$ on C is given by a Weyl transformation
- We choose to work at generic $u \in \mathcal{B}$, then the covering π has only square-root type branch points. Moreover each branch point is associated with a Weyl reflection, hence with a positive root $\alpha \in \Phi_+$.
- Above a square-root branch point, we have sheets colliding together pairwise

$$\mathcal{P}_\alpha = \{(i,j); \nu_j - \nu_i = \alpha\}, \qquad |\mathcal{P}_\alpha| = k_\rho \tag{89}$$

where k_ρ is a positive integer that only depends on ρ and not on α.

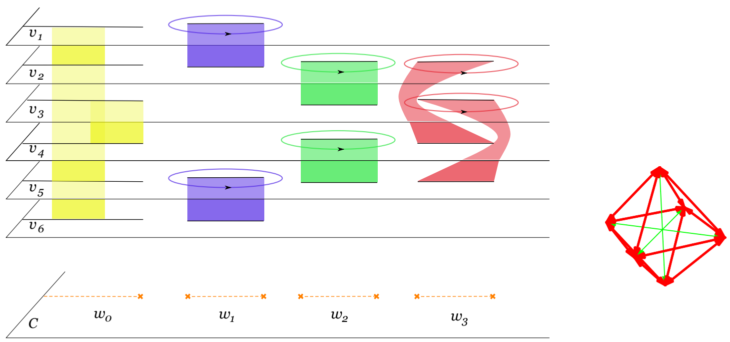

Figure 3: The six sheets of SO(6) vector cover. On the right: The weights

For example, in the 1st fundamental rep of A_n, $k_\rho = 1$. In the first fundamental rep of D_n, $k_\rho = 2$.

A simple intuition behind this point: Λ_ρ is closed under Weyl reflections, then it lies on a hypersphere in $\mathfrak{t}^*$. Two weights separated by α lie along an *affine* line parallel to α, which cuts the sphere in $0, 1, 2$ points. So the weights go at most pairwise. In fact α defines a $\mathscr{H}_\alpha$ hyperplane of orthogonal vectors to α, which cuts the sphere into 2. Each weight can be decomposed as

$$\nu = \nu_\perp + \nu_\parallel \tag{90}$$

where the $\nu_\perp$ part is common to both, while the $\nu_\parallel$ of ν_i is opposite to that of ν_j. So each pair has one weight on each hemisphere. See figure 4

- Near a ramification point of square-root type (above a branch point, in the fiber)

$$\lambda(z) \sim \lambda_0 \pm \sqrt{z}\, dz \qquad \Rightarrow \qquad \lambda_j - \lambda_i \sim \sqrt{z}\, dz \tag{91}$$

where $\lambda_0 = \langle \nu_i, \varphi(z=0) \rangle = \langle \nu_j, \varphi(z=0) \rangle$.

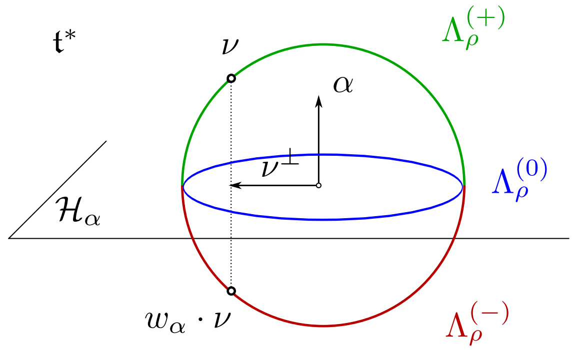

Figure 4: The splitting of the weights of aminuscule representation, induced by a choice of root α.

6.2 Definition of the network

A spectral network is a combinatorial object associated to a covering of this kind. It includes two pieces of data

- geometric data of trajectories on C
- combinatorial-topological data of (open) paths on Σ associated to each trajectory

The trajectories are called $\mathcal{S}$-walls, and are labeled by roots, they emanata from branch points, and their geometry is fixed by the equation

$$\mathcal{S}_\alpha \;:\; (\partial_t, \langle \alpha, \varphi(z) \rangle) \in e^{i\vartheta}\mathbb{R}^+ \tag{92}$$

So near a square-root branch point, we have 3 walls coming out

$$z(t) = z_0 + te^{i\frac{2}{3}(\vartheta+2\pi k)} \qquad k \in \mathbb{Z}/3\mathbb{Z} \qquad t \in \mathbb{R}^+ \tag{93}$$

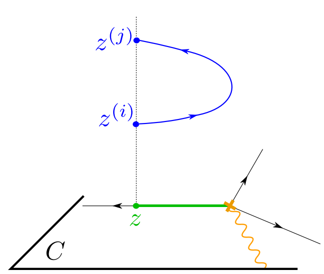

Figure 5:

As $\mathcal{S}$-walls evolve, they may intersect each other, such points are called *Joints*. If $\mathcal{S}_\alpha, \mathcal{S}_\beta$ intersect then a new wall $\mathcal{S}_{\alpha+\beta}$ will be born, if $\alpha + \beta$ is also a root.

The combinatorial data of each $\mathcal{S}$-wall is called "soliton data". To describe it we introduce the following object:

$$\Gamma_{ij}(z) = H_1^{rel}(\Sigma, (\lambda_i(z), \lambda_j(z)); \mathbb{Z}) \tag{94}$$

these are topological equivalence classes of open paths on Σ that start at $\lambda_i(z)$ and end at $\lambda_j(z)$. These spaces are torsors for $H_1(\Sigma, \mathbb{Z}) \simeq \Gamma_{e.m.}$, i.e. they are affine lattices on which there is an action by the latter.

We count these paths with integers

$$\mu(a) \qquad a \in \Gamma_{ij}(z) \qquad \mu(a) \in \mathbb{Z}\,. \tag{95}$$

The soliton content of $\mathcal{S}_\alpha$ is then a collection

$$\big\{(a, \mu(a))\,;\, a \in \Gamma_{ij}(z)\,,\; (i,j) \in \mathcal{P}_\alpha\big\} \tag{96}$$

where $z \in \mathcal{S}_\alpha$ is any point on the wall.

REMARK: The geometry of $\mathcal{S}$-walls is such that they can be lifted to very special "geodesics" on Σ. If we define the

$$\text{"mass"} \qquad M_a = \int_a |\pi^*\lambda| \tag{97}$$

$$\text{"central charge"} \qquad Z_a = \int_a \pi^*\lambda \tag{98}$$

then we see that such solitons that "lie above" S-walls are "BPS" in this sense. There is in fact a physical interpretation of these as BPS states in certain $2d\, N = (2,2)$ theories coupled to the 4d theories. We will not discuss this any further, except for a few more remarks later on.

6.3 Determination of the soliton data

The soliton data is fixed by two rules. To explain them, let me introduce a formal "homology path algebra"

$$X_a, X_\gamma \;:\quad X_\gamma X_{\gamma'} = X_{\gamma+\gamma'}, \quad X_a X_\gamma = X_\gamma X_a = X_{a+\gamma}, \quad X_a X_b = X_{ab} \text{ if } end(a) = beg(b) \tag{99}$$

- Then, the soliton content on primary walls is just *simpletons*: one soliton for each (i,j) pair. The rel.hom. class is the "simplest" path stretching from $\lambda_i(z)$ to the ramification point, and back to the $\lambda_j(z)$. The degeneracy is precisely $\mu(a) = 1$ for each of these, and zero for all other classes.

- Across intersections, the soliton content of descendant walls is determined by the solitons content of parent walls. For each wall we define the generating series

  $$\Xi_\alpha = \sum_{(i,j)\in\mathcal{P}_\alpha} \sum_{a\in\Gamma_{ij}} \mu(a) X_a \tag{100}$$

  then

  $$\Xi_{\alpha+\beta} = [\Xi_\alpha, \Xi_\beta]\,. \tag{101}$$

  The simplest incarnation of this formula is the Cecotti-Vafa wall-crossing formula, describing wall-crossing in 2d N=(2,2) theories.

This completely fixes the data of a spectral network $\mathcal{W}_\vartheta$.

6.4 The parallel transport theorem

Given a path $\wp \subset C$ (an open path), we associate to it a formal generating function, as follows.

- If $\wp \cap \mathcal{W} = \emptyset$, $F(\wp)$ is a flat diagonal parallel transport

$$F(\wp) := D(\wp) = \sum_{\nu_i \in \Lambda_\rho} X_{\wp^{(i)}} \tag{102}$$

  where $\wp^{(i)}$ is the lift of $\wp$ to the sheet $\lambda_i(z)$ corresponding to ν_i. Note that

$$F(\wp)F(\wp') = F(\wp\wp') \tag{103}$$

  by the homology path algebra rules.

- If $\wp$ does intersect an $\mathcal{S}$ wall, there is a correction

$$F(\wp) = D(\wp_+)\mathcal{S}_\alpha D(\wp_-) \qquad \mathcal{S}_\alpha = 1 + \Xi_\alpha = e^{\Xi_\alpha} \tag{104}$$

Then, the soliton content of streets implies the *Parallel transport theorem:*

$$F(\wp) \text{ is flat.} \tag{105}$$

In particular, it depends only on the homotopy class of $\wp$, and extends smoothly across branch points and joints.

REMARK: Note that $\mathcal{S}_\alpha$ looks pretty much like a transition function. We build E_z by taking the pushforward $\pi_*(\mathcal{L})$ at generic loci, away from $\mathcal{W} \subset C$. These are contractible patches and we get trivializable "bundle pieces". But then, we glue these pieces together across S-walls, with transition functions determined by the soliton content.

In this way, a spectral network provides a *Nonabelianization map*

$$\Psi_{\mathcal{W}} : \mathcal{M}_F(GL(1,\mathbb{C}),\Sigma) \to \mathcal{M}_F(GL(k,\mathbb{C}),C) \tag{106}$$

as promised.

6.5 K-wall jumps and BPS states

We still haven't got to the BPS spectrum, we do this now.

- As we change ϑ, the network geometry varies smoothly. But there are special values of $\vartheta = \vartheta_c$ for which the topology of $\mathcal{W}$ jumps discontinuously.

- When this happens, *finite $\mathcal{S}$-walls appear* stretching between two or more branch points, see figure 6

- Since the soliton data is determined by the *topology of* $\mathcal{W}$, it will jump too. The nonabelianization map, then, which depends on the soliton data, also jumps. This is described in a precise manner

$$\Psi_{\mathcal{W}(\vartheta+)} = \mathcal{K}(\Psi_{\mathcal{W}(\vartheta_-)}) \tag{107}$$

  which stands for the universal substitution rule

$$\mathcal{K}(X_a) = X_a \prod_{\gamma \in \Gamma_c} (1 - X_\gamma)^{\langle a, L(\gamma)\rangle} \tag{108}$$

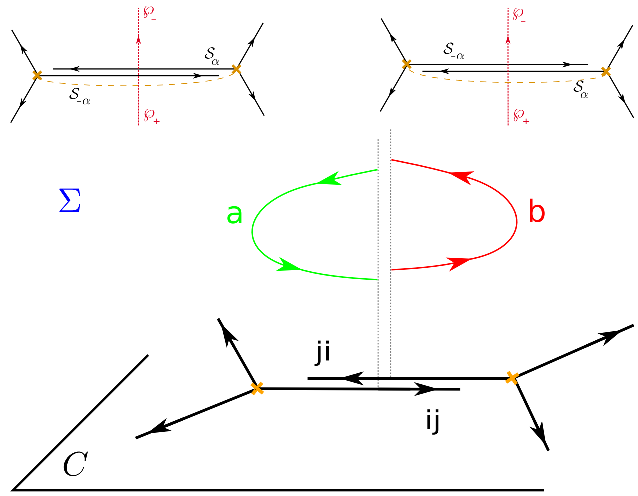

Figure 6:

- Here $\gamma \in \Gamma_c$ is a 1-dimensional sub-lattice of closed cycles, which appear as lifts of the network $\mathcal{W}$. The lattice is 1-dimensional by genericity, because $\vartheta_c = arg(Z_{\gamma_c}(u))$.
- The object $L(\gamma)$ is a 1-cycle on Σ built out of the combinatorics of soliton data for S-walls involved in the "finite network" that appears. Its homology class is proportional to γ and in fact
$$\Omega(\gamma, u) = \frac{[L(\gamma)]}{\gamma} \tag{109}$$
- The jumps are of course the same transformations that appear in the KSWCF. Scanning over $\vartheta \in (0, 2\pi)$ we get the full KS monodromy
$$\mathbb{S}_u =: \prod_\gamma \mathcal{K}_\gamma^{\Omega(\gamma,u)} : \tag{110}$$

  hence the whole BPS spectrum

For explicit examples see the following websites and papers:
`http://het-math2.physics.rutgers.edu/loom/` Draw your own spectral network, and play with examples.
`https://www.ma.utexas.edu/users/neitzke/spectral-network-movies/` movies of spectral networks and their jumps.
`http://arxiv.org/abs/1204.4824` for some simple and some advanced examples
`http://arxiv.org/abs/1305.5454` for some more advanced examples and how to get algebraic equations from spectral networks (herds)
`http://arxiv.org/abs/1601.02633` for some examples of D_n and E_n types.

Finally, there is a story of Spectral Networks "with spin" which introduced a refined counting of paths, by regular homotopy, and leads to the motivic version of the Kontsevich-Soibelman formula. This is developed here `http://arxiv.org/abs/1408.0207`.

Sources

Gaiotto, Moore, Neitzke: ”Four dimensional wall-crossing via three dimensional field theory”
Gaiotto, Moore, Neitzke: ”Wall Crossing, Hitchin systems and the WKB approximation”
Gaiotto, Moore, Neitzke: ”Framed BPS states”
Gaiotto, Moore, Neitzke: ”Wall Crossing in 2d-4d coupled systems”
Gaiotto, Moore, Neitzke: ”Spectral Networks”
Gaiotto, Moore, Neitzke: ”Spectral Networks and snakes”
Longhi, Park: ” ADE Spectral Networks”

References

See references within the sources.