Notes on ADHM construction

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Contents

1 The Ward Correspondence

$$\text{ASD connections} \xrightarrow{\text{Ward}} \text{Hol VB on } \mathbb{P}^3 \xrightarrow{\text{ADHM}} \text{Mondas on } \mathbb{P}^3 \xrightarrow{\text{Donaldson}} \text{Stable VB on } \mathbb{P}^2.$$

1.1 An integrability criterium

Consider two vector bundles $E \xrightarrow{p} X$, $F \to X$. Let D be a linear differential operator $E \to F$ of the form

$$D = \sigma\nabla,$$

where σ is the symbol homomorphism $T^*X \otimes E \to F$. We make a technical assumption, let $I_x \subset F_x$ consisting of $(Ds)_x$ for all local sections s of E, and we assume that the rank of I_x is independent of x. The operator D induces a map

$$J_1(E) \to F,$$

where $J_1(E)$ is the first jet bundle. We let R be the kernel $0 \to R \to J_1(E) \to F$. Then by our assumption of the rank of I_x, one sees that R is also of constant rank and so a subbundle of $J_1(E)$.

A section s of E can be thought of as a function $s^\vee : E^* \to \mathbb{C}/\mathbb{R}$ which is fibrewise linear: let e_a be a local basis of E, and λ_a be the fibre coordinate of E^*, then

$$s^\vee(x, \lambda) = s(x)^a \lambda_a. \tag{1}$$

The differential of $s^\vee$

$$ds^\vee = d(s^a\lambda_a) = ds^a\lambda_a + s^a d\lambda_a$$

factors thought the 1-jet $j_1(s)$ and so $ds^\vee$ gives a map of bundles over E^*

$$V:\ p^*J_1(E) \to T^*E^*.$$

The map V is a surjection when $\lambda \neq 0$, The image of R under this map will be denoted as $V(D)$.

Put less formally $V(D)$ is spanned by the following 1-forms

$$\{ds^\vee | Ds = 0\},$$

where $Ds = 0$ is solved in the first order neighbourhood of each point.

To be more specific, there are two ways $Ds = 0$, either $\nabla s = 0$, or $\nabla s \in \ker\sigma$. In the first case, $ds^a + \Gamma^a{}_b s^b = 0$, then

$$ds^\vee = d(s^a\lambda_a) = ds^a\lambda_a + s^a d\lambda_a = -\Gamma^a{}_b s^b\lambda_a + s^a d\lambda_a = \theta_a s^a.$$

But since s^a is arbitrary (any given value of s at a point, one can extend it to a section with $Ds = 0$ in the first order neighbourhood), these $ds^\vee$ are generated by θ_a. In the second case, we choose a basis for $\ker\sigma$

$$\sigma_i^{Ia} e_a dx^i \in \ker\sigma.$$

So the subbundle $V(D)$ of T^*E^* is generated by the set θ_a and $\sigma^I = \sigma_i^{Ia}\lambda_a dx^i$. To check that the distribution $\ker V(D) \subset TE^*$ is integrable, we need to show that

$$d\theta_a = (\cdots)\theta_b + (\cdots)\sigma^I, \quad d\sigma^I = (\cdots)\theta_b + (\cdots)\sigma^J.$$

To wit

$$\begin{aligned} d\theta_a &= -\Omega^b{}_a\lambda_b - \Gamma^b{}_a\theta_b, \\ d\sigma^I &= d\sigma_i^{Ia}\cdot\lambda_a dx^i + \sigma_i^{Ia} d\lambda_a \cdot dx^i = \nabla\sigma_i^{Ia}\cdot\lambda_a dx^i + \sigma_i^{Ia}\theta_a\cdot dx^i \end{aligned}$$

where $\nabla\sigma_i^{Ia}$ is only covariant in the a index. So we get that Ω and $\nabla\sigma_i^{Ia}$ must be in the ideal generated by θ, σ.

Example Let now the symbol map be the identity, i.e. $D = \nabla$, then the set σ^I is empty and

$$\ker V(\nabla) = \ker\langle\theta_a\rangle = \langle\partial_{x^i} + \Gamma^a_{ib}\lambda_a\partial_{\lambda^b}\rangle,$$

which as we know is integrable iff the curvature is zero.

On the other hand the condition on the curvature says $\Omega^b{}_a\lambda_b = f_a^{bc}\theta_c\lambda_b$, which is impossible since Ω will never have $d\lambda$.

Example Let now the symbol be the projection to the (0,1) component, i.e. $F = E \otimes T^{0,1}X$. Then the kernel of the symbol map is $\sigma_a^i = \lambda_a dx^i$, and

$$\frac{\partial}{\partial x^{\bar{i}}} + \Gamma^b_{\bar{i}a}\lambda_b\frac{\partial}{\partial\lambda_a}. \tag{2}$$

The integrability condition for $d\sigma^i_a$ leads to

$$d\sigma^i_a = d\lambda_a dx^i = \theta_a dx^i + \Gamma^b_{ma}\lambda_b dx^m dx^i$$

which is already in the ideal generated by γ and σ. The condition on curvature says that its $(0,2)$ part is zero, since its (1,1) and $(2,0)$ part $\Omega_{i\bar{j}}{}^b{}_a\lambda_b dx^i dx^{\bar{j}}$ or $\Omega_{ij}{}^b{}_a\lambda_b dx^i dx^j$ is in the ideal generated by σ. So we get the familiar condition that Eq.2 gives a Cauchy-Riemann operator on E^* if the (0,2) part of the curvature is zero

1.2 Application to the spin bundle

Now the *conformal Killing spinor* equation

$$D: \ S_+ \to S_+ \otimes T^*M, \quad D\psi = \nabla\psi - \frac{1}{4}\Gamma_i \nabla\!\!\!/\psi \otimes dx^i. \tag{3}$$

Note that now the symbol map is the projection to the kernel of the Clifford multiplication: slash

$$\text{slash}: \quad \psi_a \otimes dx^i - \frac{1}{4}\Gamma_j\Gamma^i\psi_a \otimes dx^j \ \to \ \Gamma^i\psi_a - \frac{1}{4}\Gamma^j\Gamma_j\Gamma^i\psi_a = 0.$$

The kernel of this symbol is in fact S_-, by the map σ^*

$$\xi \in S_-, \quad \sigma^*\xi = \frac{1}{4}\Gamma_i\xi \otimes dx^i \in S_+ \otimes T^*M.$$

One then expresses $\Gamma_i\xi$ in terms of the basis $\{e_a\}$ for S_+

$$\Gamma_i\xi = \langle \check{e}^a, \Gamma_i\xi\rangle e_a,$$

where $\langle,\rangle$ is a pairing (it seems more appropriate to use Majorana pairing) and $\check{e}^a$ is the dual basis associated with the paring. Then $V(D)$ is locally spanned by (replacing e_b with λ_b)

$$\theta_a = d\lambda_a - \Gamma^b{}_a\lambda_b, \quad \sigma_\xi = \langle \check{e}^b, \Gamma_p\xi\rangle\lambda_b \otimes dx^p,$$

where λ_a is the coordinate of $(S_+)^*$, and ξ is a section of S_-. The above equation must be considered as in $T^*_{\mathbb{C}}E^*$, since the spinnors are naturally complex (so λ_a is a holomorphic coordinate) and σ_a will be complex forms. We look at the kernel of σ_ξ for all ξ. At a point (x,λ), the kernel of σ_ξ is $X^i\partial_i$ such that

$$\langle \check{e}^b, X\!\!\!/\xi\rangle\lambda_b = 0, \quad \forall \xi \in S_-.$$

So X is the annihilator of the spinor $\lambda_a e^a$, which determines a half rank subbundle of $T^*_{\mathbb{C}}E^*$ with

$$\ker\langle\sigma_a\rangle \cap \overline{\ker\langle\sigma_a\rangle} = 0,$$

since if $X\!\!\!/\cdot e^a\lambda_a = 0$, then $X^2 = 0$ and X must be complex with nonzero real and imaginary parts. From this we also get $\ker\langle\sigma_a\rangle \oplus \overline{\ker\langle\sigma_a\rangle}$ covers the entire horizontal part of $T^*_{\mathbb{C}}E^*$.

We check again the integrability condition (where ξ_a is a basis of S_-)

$$d\sigma_a = \check{e}^b\Gamma_p\xi_a\theta_b \otimes dx^p + \check{e}^b\Gamma_p\xi_a\Gamma^c{}_b\lambda_c \otimes dx^p + \check{e}^b\Gamma_p(d\xi_a)\lambda_b \otimes dx^p,$$

we replace $d\xi_a = \nabla\xi_a - \Gamma^b{}_a\xi_b$ and the two terms with spin connections combine into the LC acting on the p index, $\Gamma^r_{qp}\check{e}^b\Gamma_r\xi_a\lambda_b \otimes dx^q \wedge dx^p$, which vanishes since it is torsionless.

$$d\sigma_a = \check{e}^b\Gamma_p\xi_a\theta_b \otimes dx^p + \check{e}^b\Gamma_p(\nabla\xi_a)\lambda_b \otimes dx^p.$$

This shows $d\sigma_a$ is in the ideal $\langle\theta_a,\sigma_a\rangle$

While the condition on the curvature can be satisfied if Ω is self-dual, since

$$\Omega\psi_+ = \Omega\gamma_5\psi_+ = -*\Omega\psi_+ = -\Omega\psi_+,$$

where $\gamma_5 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4$ is the chirality operator. From the above discussion we have a half rank subbundle $\ker V(D)$ of $T_{\mathbb{C}}E^*$, depending on the fibre of S_+, such that

$$\ker V(D)\cap\overline{\ker V(D)} = 0$$

and satisfies the integrability criteria above, i.e. we have an integrable complex structure.

Lemma 1.1 *The operator*

$$D\psi - \frac{1}{4}\Gamma_p \not{D}\psi\otimes dx^p,$$

transforms homogeneously under the conformal transformation $\tilde{g} = e^{2\phi}g$*, if* ψ *transforms as* $\tilde{\psi} = e^{1/2\phi}\psi$*.*

Proof Under the conformal transformation, the spin connection changes as

$$\begin{aligned}
\tilde{\omega}_{ij} &= \tilde{g}_{\mu\nu}\tilde{e}_i^\mu\tilde{\nabla}\tilde{e}_j^\nu = \tilde{g}_{\mu\nu}\tilde{e}_i^\mu\big(-d\phi\tilde{e}_j^\nu + e^{-\phi}\tilde{\nabla}e_j^\nu\big) \\
&= -d\phi\delta_{ij} + \tilde{g}_{\mu\nu}\tilde{e}_i^\mu e^{-\phi}\big(\nabla e_j^\nu + d\phi e_j^\nu + (e_j\cdot\partial\phi)dx^\nu - (\partial^\nu\phi)e_{j\rho}dx^\rho\big) \\
&= -d\phi\delta_{ij} + \omega_{ij} + d\phi\delta_{ij} + e_{[i}(e_{j]}\circ\phi) = \omega_{ij} + e_{[i}(e_{j]}\circ\phi).
\end{aligned}$$

So the spin derivative goes to ($\tilde{\psi} = e^{w\phi}\psi$)

$$\delta\tilde{D}\tilde{\psi} = w\tilde{\psi}\otimes d\phi + \frac{1}{4}\Gamma^{ij}e_{[i}(e_{j]}\circ\phi)\tilde{\psi} = w\tilde{\psi}\otimes d\phi + \frac{1}{2}\Gamma_p(d\phi\cdot\Gamma)\tilde{\psi}\otimes dx^p - \frac{1}{2}\tilde{\psi}\otimes d\phi.$$

And so

$$\delta\tilde{D}\tilde{\psi} - \frac{1}{4}\Gamma_p\tilde{\not{D}}\otimes dx^p = w\tilde{\psi}\otimes d\phi + \frac{1}{2}\Gamma_p(d\phi\cdot\Gamma)\tilde{\psi}\otimes dx^p - \frac{1}{2}\tilde{\psi}\otimes d\phi - \frac{1}{4}(\frac{3}{2}+w)\Gamma_p(d\phi\cdot\Gamma)\tilde{\psi}\otimes dx^p,$$

so if $w = 1/2$, the inhomogeneous terms vanish ∎

From this we see that the complex structure over $P(S_+)$ only depends on the conformal class of the metric of X.

Example We will investigate $P(S)$ of $\mathbb{P}^2$. Now taking the projective version of S is crucial, since $\mathbb{P}^2$ is not spin. We can take the spin bundle to be

$$S = \Omega^{0,\bullet}(\mathbb{P}^2) = S_+\oplus S_- = \Omega^{0,1}(\mathbb{P}^2)\oplus\Omega^{0,\text{even}}(\mathbb{P}^2).$$

How did we determine the chirality here? The gamma matrices are

$$\Gamma^z = 2\iota_{\bar{z}}, \quad \Gamma^{\bar{z}} = 2d\bar{z}$$

that is

$$\begin{aligned}
\Gamma^x &= (\iota_{\bar{z}} + d\bar{z}), \quad \Gamma^y = -i(\iota_{\bar{z}} - d\bar{z}) \\
\Gamma^x\Gamma^y &= -2i\deg + i.
\end{aligned}$$

Thus $\Gamma^1 \cdots \Gamma^4$ acting on 1-forms $d\bar{z}^{1,2}$ gives $+1$.

Let the inhomogeneous coordinates in one patch be $[z_1, z_2, 1]$, and $[y_1, 1, y_3]$ another. We have the change of trivialisation

$$dy_1 = d(z_1/z_2) = z_2^{-1}dz_1 - z_2^{-2}z_1 dz_2, \quad dy_3 = d(1/z_2) = -z_2^{-2}dz_2,$$
$$\begin{bmatrix} dy_1 \\ dy_3 \end{bmatrix} = \begin{bmatrix} z_2^{-1} & -z_2^{-2}z_1 \\ 0 & -z_2^{-2} \end{bmatrix} \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}. \tag{4}$$

On the other hand, we look at the full flag variety F_3, and we have a clear fibration $F_3 \to \mathbb{P}^2$ and we will determine the transition function of the fibre. The plane passing $\ell = [z_1, z_2, 1]$ is spanned by ℓ and u, which can be chosen to be $u = [a, b, 0]$. In a different patch $\ell = [y_1, 1, y_3]$, we let $v = [c, 0, d]$. If $\ell,\ v$ are to span the same plane

$$[a, b, 0] \sim \left[a - \frac{bz_1}{z_2}, 0, -\frac{b}{z_2}\right],$$
$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 & -z_1 z_2^{-1} \\ 0 & -z_2^{-1} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

This transition function agrees with Eq.4 up to a scalar, so we conclude that

$$P(S_+) \simeq P(F_3),$$

considered as bundle over $\mathbb{P}^2$.

2 Geometry of S^4

Recall that S^4 is in fact the quaternion projective space

$$S^4 \simeq \mathbb{H}P^1$$

realised as follows. Let $q_1,\ q_2$ be a pair of quaternions. Impose the condition $|q_1|^2 + |q_2|^2 = 1$, we get S^7. And we can further mod out by the unit quaternions, i.e. $SU(2) \simeq S^3$, so we get the fibration $S^3 \to S^7 \to S^4$. If we first mod out by $U(1)$, we get the fibration

$$\begin{array}{ccccc} SU(2)/U(1) & \longrightarrow & S^7/U(1) & \longrightarrow & S^7/SU(2) \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ \mathbb{C}P^1 & \longrightarrow & \mathbb{C}P^3 & \longrightarrow & S^4 \end{array}, \tag{5}$$

We parameterise S^4 as $\mathbb{H}P^1$, i.e. $[q_1, q_2]$, with $q_1 = x + jy$, and $q_2 = z + jw$. In this way[1] of writing $q_{1,2}$, one has $\mathbb{H}^2/\mathbb{C}^* \simeq \mathbb{P}^3$. The parameterisation of this quotient can be chosen as

$$[q_1, q_2]_{\mathbb{C}^*} = (q_1, q_2) \times_{\mathbb{H}^*} \mathbb{H}/\mathbb{C}^* \tag{6}$$

e.g. in the patch $q_2 \neq 0$, one has $(q_1, q_2) \to (q_1 q_2^{-1}, 1) \times [q_2]_{\mathbb{C}^*}$, and this parameterisation emphasises the fibration Eq.5. And one sees that $\mathbb{P}^3$ is the associated bundle with fibre $SU(2)/U(1) \sim S^2$

$$S^7 \times_{SU(2)} SU(2)/U(1) \sim S^7/U(1) \sim \mathbb{P}^3.$$

[1] It is important to write $x + jy$ instead of $x + yj$

Proposition 2.1 *The bundle of Eq.6, i.e.*

$$V = [q_1, q_2] \times_{\mathbb{H}^*} \mathbb{C}^2 \to [q_1, q_2]_{\mathbb{H}^*} = \mathbb{H}P^1$$

is the right-handed spin bundle S_+ over S^4.

??The total space of the right handed spin bundle S_+ over S^4 is $\mathcal{O}(-1)$ over $\mathbb{P}^3$.??

Proof On S^4, over the patch $U_1: \; q_2 \neq 0$, we trivialise the above bundle as[2]

$$V|_{U_1}: \; [q_1 q_2^{-1}, 1] \times \psi, \quad \psi = \psi_1 + j\psi_2$$

where $\psi_{1,2}$ are the fibre coordinates of V. Over $U_2: \; q_1 \neq 0$, we trivialise V as $[1, q_2 q_1^{-1}] \times \tilde{\psi}$. In this convention, the structure group $SU(2)$ acts from the left as (where $q = q_1 q_2^{-1}$)

$$[q, 1] \times \psi = [1, q^{-1}] \times \tilde{\psi} \;\Rightarrow\; q\psi = \tilde{\psi}. \tag{7}$$

From Eq.7 we see that the transition function $U_1 \cap U_2 \to SU(2)$ is the identity map, and hence of degree 1. Since any principal G bundle over S^4 is classified by the map $S^3 \to G$, we have completely fixed the bundle V. We need to compare this with the spin bundle of S^4.

Embed S^4 in $\mathbb{R}^5$ in the standard way, let M_{ij} be the standard generators of $\mathfrak{so}(5)$ (e.g. $M_{12} = i\sigma_2$), and $[M_{12}, M_{23}] = M_{13}$. At the north pole choose the standard frame of S^4 by letting the normal be the 4^{th} direction, while the tangent vectors of S^4 occupy the $0,1,2,3$ directions. The rotation from the north pole along the direction $\hat{r}$ by θ degrees is

$$\left[\begin{array}{cc} 1 - \hat{r}\otimes\hat{r} + \cos\theta\hat{r}\otimes\hat{r} & \sin\theta\hat{r} \\ -\sin\theta\hat{r} & \cos\theta \end{array}\right].$$

At the south pole choose the frame $[1 - 2r_0 \otimes r_0, -1]$, with $r_0 = [1,0,0,0]$. Transporting this frame to the rest of the sphere gives a framing except at the north pole. Draw a circle round the north pole parameterised by $\hat{r}$, the transition function (of the coordinate of the tangent bundle) is thus (the following multiplies the coordinate of the north patch and gives that of the south patch, matching the convention used earlier)

$$g(\hat{r}) = (1 - 2r_0 \otimes r_0)^{-1}(1 - 2\hat{r}\otimes\hat{r}) \in SO(4)$$

If $\tilde{g}(\hat{r})$ is the lift, then to lift $g(\hat{r} + \delta\hat{r})$, we need to lift

$$g^{-1}\delta g = -2(1 - 2\hat{r}\otimes\hat{r})^{-1}\delta(\hat{r}\otimes\hat{r}) = -2(\delta\hat{r}\otimes\hat{r} - \hat{r}\otimes\delta\hat{r}) \quad\Rightarrow\quad \hat{g}^{-1}\delta\hat{g} = (\hat{r}\cdot\Gamma)(\delta\hat{r}\cdot\Gamma).$$

where the Lorentz generators are $M_{ij} = 1/2\Gamma_i\Gamma_j$. This can then be integrated to give the lift (fixing $\hat{g}(r_0) = 1$), as follows. Pick (arbitrarily)

$$\Gamma_0 = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right], \quad \Gamma_i = \left[\begin{array}{cc} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{array}\right], \quad \gamma = \Gamma_0\Gamma_1\Gamma_2\Gamma_3 = \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right], \quad \hat{r}\cdot\Gamma = \left(\begin{array}{cc} 0 & \bar{q} \\ q & 0 \end{array}\right). \tag{8}$$

Then

$$(\hat{r}\cdot\Gamma)(\delta\hat{r}\cdot\Gamma) = \left[\begin{array}{cc} \bar{q}\delta q & 0 \\ 0 & q\delta\bar{q} \end{array}\right] \;\Rightarrow\; \hat{g}(\hat{r}) = \left(\begin{array}{cc} q & 0 \\ 0 & \bar{q} \end{array}\right).$$

While from the expression of γ, the upper entry has $+1$ chirality ∎

[2] the tautological bundle over $\mathbb{P}^n$ is $\mathcal{O}(-1)$, the way to remember this is that it has no global section.

We will now analyze the spin bundle more closely. A coordinate of the total space of the spinors S_+ is chosen $(q_1, q_2) \to [q_1 q_2^{-1}, 1] \times q_2$, and here one regards q_2 as a 2-vector $[z, w]$. The action of a quaternion on $u + jv$ is

$$(x + jy)(u + jv) = xu + j\bar{x}v + jyu - \bar{y}v, \quad (x + jy)(u + jv) \sim \begin{bmatrix} x & -\bar{y} \\ y & \bar{x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{9}$$

At each point of the fibre of S_+, i.e. given each spinor $\psi = [u, v]^T$, the subbunble of $T_{\mathbb{C}} S^4$ which annihilates ψ defines a holomorphic structure[3]

Proposition 2.2 *The complex structure determined as above plus the standard complex structure of the fibre coincides with the standard complex structure of* $\mathbb{P}^3$

Proof The annihilator of $\psi = [z, w]$ are those δq such that $\delta q(z + jw) = 0$. We work now in the patch $\{w \neq 0\} \subset \mathbb{P}^3$. We set $w = 1$, so x, y, z are the inhomogeneous coordinates of $\mathbb{P}^3$. Since $q = (x+jy)(z+j)^{-1}$, we have $\delta q = (\delta x + j\delta y)(z+j)^{-1} - (x+yj)(z+j)^{-1}\delta z(z+j)^{-1}$, and the action of δq on the spinor $\psi = [z, 1]^T$ is

$$\begin{aligned} \delta q \cdot \psi &= \delta q(z + j) = (\delta x + j\delta y) - (x + jy)(z + j)^{-1}\delta z \\ &= \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} - \frac{1}{|z + j|^2} \begin{bmatrix} (x\bar{z} + \bar{y})\delta z \\ (y\bar{z} - \bar{x})\delta z \end{bmatrix}. \end{aligned}$$

Here the key is that the first column of the matrix in Eq.9 is holomorphic in x, y. Now that $\delta q \psi$ only contains $\delta x, \delta y, \delta z$, and if $J\delta x/y/z = i\delta x, y, z$, then $(1 + iJ)\delta q$ annihilates ψ ■

3 The ADHM construction

We work on $\mathbb{C}^2$ or S^4. The geometrical facts of S^4 used here are recalled in sec.2. Locally looking at $\mathbb{R}^4$ parameterised by $q = q_0 + iq_1 + j(q_2 + iq_3)$, one has three complex structures induced by right multiplying i, j, k

$$\begin{aligned} I(q_0, q_1, q_2, q_3) &= (-q_1, q_0, -q_3, q_2), \\ J(q_0, q_1, q_2, q_3) &= (-q_2, q_3, q_0, -q_1), \\ K(q_0, q_1, q_2, q_3) &= (q_3, q_2, -q_1, -q_0). \end{aligned}$$

With the volume form $dq_0 \cdots dq_3$, the following forms are anti-self-dual

$$\omega_1 = dq_0 dq_1 - dq_2 dq_3, \quad \omega_2 = dq_0 dq_2 + dq_1 dq_3, \quad \omega_3 = dq_0 dq_3 - dq_1 dq_2,$$

and one important feature of these forms are that they are (1,1) w.r.t all three complex structures, e.g.

$$\begin{aligned} I(\omega_3) &= I(dq_0 dq_3 - dq_1 dq_2) = (-dq_1)dq_2 - dq_0(-dq_3) = \omega_3, \\ I(\omega_1) &= I(dq_0 dq_1 - dq_2 dq_3) = (-dq_1)dq_0 - (-dq_3)dq_2 = \omega_1. \end{aligned}$$

[3] which changes along the fibre of the spin bundle. This is the main novelty, recall that S^4 itself does not have even an almost complex structure.

While in contrast for, say, $\omega = dq_0dq_1 + dq_2dq_3$

$$J(dq_0dq_1 + dq_2dq_3) = (-dq_2)dq_3 + dq_0(-dq_1) = -\omega_1,$$
$$I(dq_0dq_1 + dq_2dq_3) = (-dq_1)dq_0 + (-dq_3)dq_2 = \omega_1.$$

Now over each point of S^4 we have a family of complex structures parameterised by $\mathbb{P}^1$, so if the curvature of a VB on S^4 lifts to (1,1) on $\mathbb{P}^3$, it is (1,1) w.r.t to I, J, K and as we saw above, must be anti-self-dual. To conclude looking for ASD solutions amounts to looking for holomorphic VB over $\mathbb{P}^3$ that is trivial over $\mathbb{P}^1$.

3.1 Horrocks construction

We start from an

Example The vector space $\mathbb{C}^4 \sim \mathbb{H}^2$ is equipped with the standard Hermitian inner product and an automorphism σ: right multiplication by j, and the combo gives a holomorphic symplectic form $\omega = \langle \sigma -, - \rangle$

$$\langle \sigma[x, y, z, w], [x', y', z', w'] \rangle = \langle [-\bar{y}, \bar{x}, -\bar{w}, \bar{z}], [x', y', z', w'] \rangle = -yx' + xy' - wz' + zw'.$$

Let a point $a \in \mathbb{P}^3$, represented as a line $L_a \subset \mathbb{C}^4$. The symplectic orthogonal of L_a, denoted $L_a^\perp$, is of dimension 3 and contains L_a. Thus

$$E = L_a^\perp / L_a$$

is a rank two bundle over $\mathbb{P}^3$. But we need to make sure E is trivial along the fibre $\mathbb{P}^1 \to \mathbb{P}^3 \to S^4$. Let a line in $\mathbb{P}^3$ be represented by a plane $\langle L_a, L_b \rangle \subset \mathbb{C}^4$, then $R = L_a^\perp \cap L_b^\perp$ has dimension 2 except $L_a \| L_a$ which is not possible.

If $\omega(L_a, L_b) \neq 0$, then R provides the trivialisation of L_z for $z \in \overline{ab}$, indeed, $\omega(R, \lambda L_a + \mu L_b) = 0$ so $R \subset E_z$. Further Since $\omega(L_z, L_{a,b}) = \omega(\lambda L_a + \mu L_b, L_{a,b})$ cannot both be be zero so $L_z \nsubseteq R$ and hence the map $R \to E_z$ is into and thus R trivialises E.

While if $\omega(L_a, L_b) = 0$, then we can describe R as $\alpha L_a + \beta L_b$ which is clearly 2D. But now the vector $\alpha L_a + \beta L_b$ will be zero precisely at one point $\alpha/\beta = \lambda/\mu$. Thus E has a section with one zero when restricted to a sub-$\mathbb{P}^1$, and hence cannot be $\mathcal{O} \oplus \mathcal{O}$, i.e. not trivial.

On a real line $\overline{a\sigma(a)}$ which corresponds to a point of S^4, we check

$$\omega(L_{\sigma a}, L_a) = \langle \sigma\sigma a, a \rangle = -||a||^2 \neq 0,$$

thus E is trivial when restricted to the fibre $\mathbb{P}^1 \to \mathbb{P}^3 \to S^4$. In fact $L_{\sigma a}^\perp = (\sigma L_a)^\perp$ is the subspace orthogonal to L_a under $\langle -, - \rangle$, and $L_a^\perp \cap L_{\sigma a}^\perp$ is invariant under σ and hence obtains a Hermitian structure. This way we obtain a holomorphic $SU(2)$ bundle trivial over each fibre and hence an instanton.

Next we generalise this example. Consider first a vector space V of dimension $2k + 2$ with a holomorphic symplectic form ω and W is a vector space of dimension k. Consider a map A depending linearly on the homogeneous coordinate $[x, y, z, w]$ of $\mathbb{P}^3$, and we demand that the image of A is of rank k and isotropic i.e. $\omega(A(W), A(W)) = 0$ at any point. Now since $E = A(W)^\perp / A(W)$ is of dimension $k + 2 - k = 2$ and we have a vector bundle.

The analogue of $L_a \cap L_b = 0$ for a line $\overline{ab}$ in the previous example is that

$$A_a(W) \cap A_b(W) = 0.$$

To show this, we note that the image of A is isomorphic to $W[-1]$ since A is assumed to be injective at every point. If there is $\xi = A_a(W) \cap A_b(W)$ then it gives a nontrivial section of $W[-1]$ over $\mathbb{P}^1$ tensored with a bundle over the Grassmannian of $Gr(V,k)$ whose fibres over a k-subspace U, are maps from $U^\perp \to U$, which is not possible.????

To show that the bundle is trivial over the Hopf fibre, take a line $\overline{ab}$, then $R = A_a(W)^\perp \cap A_b(W)^\perp$ has dimension 2 since $A_a(W) \cap A_b(W) = 0$. Again $\omega(R, \lambda A_a x + \mu A_b x) = 0,\ \forall x$, and $R \cap A_{\lambda a+\mu b}(W) = 0$ if $\omega(A_a(W), A_b(W)) \neq 0$. So the jumping lines are those $\overline{ab}$ with $\omega(A_a(W), A_b(W)) = 0$.

As we are now in the holomorphic setting, the structure group of E is $Sl(2,\mathbb{C})$ (since E inherits a symplectic structure), to obtain an $SU(2)$ bundle we need to impose a real structure. We demand that $A_{\sigma a} = \sigma A_a$, then on a real line $\overline{a\sigma(a)}$ which corresponds to a point of S^4, we check if it can be a jumping line

$$\omega(A_a(W), A_{\sigma a}(W)) = \omega(A_a(W), \sigma A_a(W)) = \langle A_a(W), A_a(W)\rangle \neq 0.$$

In fact over a real line R can be described as $V = A_a(W) \oplus A_{\sigma a}(W) \oplus R$, i.e. regarding R as the orthogonal complements of $A_a(W)$ and $A_{\sigma a}(W)$. Clearly R is invariant under σ and has a Hermitian structure.

3.2 Monad construction

We still work on $\mathbb{C}^2$ or S^4.

We generalise the construction of the previous section. Recall the key ingredient above is a map $A : W[-1] \to V$ whose image is isotropic. We can dualise the map $A^* :\ V^* \to W^*[+1]$ and use the symplectic structure to rewrite the dual map as $A^* :\ V \to W[+1]$. The isotropy says that $A^* \circ A = 0$. Now we let A^* be any map $B :\ V \to W[+1]$, thus we arrive at the monad

$$\mathscr{E} :\ \ W^k[-1] \overset{A_X}{\hookrightarrow} V^{2k+\ell} \overset{B_X}{\twoheadrightarrow} W^k[+1], \quad B_X \circ A_X = 0 \tag{10}$$

that is E arises as the homology at the middle of the monad. The ranks of $W\ V$ are $k, 2k+\ell, k$ respectively.

We also want to consider *framed instatons*, i.e. we fix the trivialisation of E at the locus $z = w = 0$. So up to a $Gl(k)$ transformation we can narrow down to (we use a subscript to denote the corresponding component)

$$A_x = \begin{bmatrix} 1_k \\ 0_k \\ 0_\ell \end{bmatrix}, \quad A_y = \begin{bmatrix} 0_k \\ 1_k \\ 0_\ell \end{bmatrix}, \quad B_x = [0_k, 1_k, 0_\ell], \quad B_y = [-1_k, 0_k, 0_\ell].$$

The other components $A_{z,w}$, $B_{z,w}$ are not fixed, so we just write

$$A = \begin{bmatrix} x + z\alpha_1 + w\hat\alpha_1 \\ y + z\alpha_2 + w\hat\alpha_2 \\ za + w\hat a \end{bmatrix}, \quad B = \Big[-y - z\alpha_2 - w\hat\alpha_2, x + z\alpha_1 + w\hat\alpha_1, zb + w\hat b\Big].$$

In order that $B \circ A = 0$, we get

$$\begin{aligned} B \circ A \ &= \ z^2(-\alpha_2\alpha_1 + \alpha_1\alpha_2 + ba) + w^2(-\hat\alpha_2\hat\alpha_1 + \hat\alpha_1\hat\alpha_2 + \hat b\hat a) \\ &\quad + zw\big(-\alpha_2\hat\alpha_1 - \hat\alpha_2\alpha_1 + \alpha_1\hat\alpha_2 + \hat\alpha_1\alpha_2 + b\hat a + \hat b a\big) = 0, \end{aligned}$$

and we get the key equations

$$
\begin{aligned}
&[\alpha_1, \alpha_2] + ba = 0, \\
&[\hat{\alpha}_1, \hat{\alpha}_2] + \hat{b}\hat{a} = 0, \\
&[\alpha_1, \hat{\alpha}_2] + [\hat{\alpha}_1, \alpha_2] + b\hat{a} + \hat{b}a = 0.
\end{aligned} \tag{11}
$$

Now the cohomology of $\mathscr{E}$ in the middle gives a bundle over $\mathbb{P}^3$, we write the complex as

$$
K \overset{A_X^\dagger \oplus B_X}{\twoheadrightarrow} L \oplus L,
$$

and since A_X is injective so $A_X^\dagger A_X$ is positive definitive and thus $A_X^\dagger$ serves as the inverse. The kernel of $A_X^\dagger \oplus B_X$ gives us the vector bundle on $\mathbb{P}^3$. But we want a vector bundle over S^4, so the kernel must be constant along the fibre of the above fibration. This can be achieved by demanding the invariance of the kernel under the right multiplication of j (see Eq.6, invariance of right multiplication by j ensures that the cohomology of the monad descends to $S^4 = \mathbb{HP}^1$)

$$
[x + jy, z + jw]j = [j\bar{x} - \bar{y}, j\bar{z} - \bar{w}].
$$

This action prompts us to define the complex structure

$$
\sigma[x, y, z, w] = (-\bar{y}, \bar{x}, -\bar{w}, \bar{z}).
$$

The invariance can be satisfied if

$$
A_X^\dagger = -B_{\sigma X}, \quad B_X = A_{\sigma X}^\dagger
$$

that implies

$$
\begin{aligned}
&\bar{x} + \bar{z}\alpha_1^\dagger + \bar{w}\hat{\alpha}_1^\dagger = \bar{x} - \bar{w}\alpha_2 + \bar{z}\hat{\alpha}_2, \\
&\bar{y} + \bar{z}\alpha_2^\dagger + \bar{w}\hat{\alpha}_2^\dagger = \bar{y} + \bar{w}\alpha_1 - \bar{z}\hat{\alpha}_1, \\
&\bar{z}a^\dagger + \bar{w}\hat{a}^\dagger = \bar{w}b - \bar{z}\hat{b} \\
\Rightarrow \qquad &\alpha_1^\dagger = \hat{\alpha}_2, \quad \hat{\alpha}_1 = -\alpha_2^\dagger, \quad a^\dagger = -\hat{b}, \quad \hat{a} = b^\dagger.
\end{aligned}
$$

Rewriting the Eq.11 and we have

Theorem 3.1 *(ADHM) The moduli space of anti-self-dual connection is described by the quotient of a set of matrices $\alpha_{1,2} \in M_k$ and $a \in M_{lk}$, $b \in M_{kl}$, satisfying*

$$
\begin{aligned}
&[\alpha_1, \alpha_2] + ba = 0, \\
&[\alpha_1, \alpha_1^\dagger] + [\alpha_2, \alpha_2^\dagger] + bb^\dagger - a^\dagger a = 0.
\end{aligned} \tag{12}
$$

and that

$$
\begin{bmatrix} x + \alpha_1 \\ y + \alpha_2 \\ a \end{bmatrix} \textit{ is injective,} \quad \begin{bmatrix} -\alpha_2 - y & \alpha_1 + x & b \end{bmatrix} \textit{ is surjective} \tag{13}
$$

The $U(k)$ action acts by $\alpha_i \to g\alpha_i g^{-1}$, $a \to ag^{-1}$ and $b \to gb$.

Proof We will not prove the exhaustiveness of the construction, but we will explain the conditions Eq.13.

We want the maps A_X and B_X to be into and onto respectively. Consider two patches $[x, y, 1, 0]$ and $[x, y, 0, 1]$, they cover $\mathbb{P}^3$ except the line at $z = w = 0$, where the (in)surjectivity is already secured. Further A_X is injective iff $\sigma A_X = A_{\sigma X}$ is and similarly for B_X. Thus it suffices to check the in(sur)jectivity at $[x, y, 1, 0]$, and hence Eq,13 ■

3.3 Remark by Donaldson

We repeat the same monad construction on $\mathbb{P}^2$, consider again the monad Eq.10 We want the cohomology to be trivial at $\ell_\infty = \{z = 0\}$, Since this monad must be trivial (exact) in the middle along the sub $\mathbb{P}^1$, then up to a $Gl(k)$ transformation, one can fix

$$A = \begin{bmatrix} x + z\alpha_1 \\ y + z\alpha_2 \\ za \end{bmatrix}, \quad B = [-y - z\alpha_2, x + z\alpha_1, zb] \, .$$

Forcing $B \circ A = 0$, we get $[\alpha_1, \alpha_2] + ba = 0$.

To mod out the $Gl(k)$ action we need to think about stability. A point $x \in \mathbb{C}^k$ is stable under $Gl(k)$ if the action is proper and in particular the orbit $g.x$ is closed. The closedness means that we can find among $g.x$ a point with the smallest norm (let $r = |x|$, then $B_r \cap G.x$ is compact, so the minimal exists). Now for adjoint actions, $\alpha \to g\alpha g^{-1}$, if the point α is of the smallest norm, then ($||\alpha||^2 = \mathrm{Tr}[\alpha^\dagger \alpha]$)

$$0 = \delta_g ||g\alpha g^{-1}||^2\big|_{g=1} = \mathrm{Tr}[\alpha^\dagger [T, \alpha]] + \mathrm{Tr}[[T, \alpha]^\dagger \alpha], \quad \delta g = gT$$

we choose T hermitian since the anti-hermitian part belongs to $\mathfrak{u}(k)$, so

$$\mathrm{Tr}[T[\alpha, \alpha^\dagger]] = 0$$

that is $\mu(\alpha) = [\alpha, \alpha^\dagger] = 0$.

Now apply this to the action of $Gl(k)$ on $\alpha_{1,2}, a, b$

$$0 = \delta_g \big(||\alpha_1||^2 + ||\alpha_2||^2 + ||a||^2 + ||b||^2 \big) = 2\mathrm{Tr}\big[T([\alpha_1, \alpha_1^\dagger] + [\alpha_2, \alpha_2^\dagger] + bb^\dagger - a^\dagger a)\big].$$

We see that the last condition coincides with the last one of Eq.12.

Now we analyze the consequence of stability, we need

Proposition 3.2 *(Hilbert criterium) A point $x \in \mathbb{C}^k$ is stable under $Gl(k)$ action iff it is stable under all 1-parameter subgroup of the form*

$$g_t = g\, diag\, [t^{w_1}, \cdots, t^{w_k}] g^{-1},$$

with w_i not all zero. In particular, the image of x tends to ∞ as $t \to \infty$.

Using this criterium, we have

Proposition 3.3 *Assuming*

$$[\alpha_1, \alpha_2] + ba = 0,$$

the condition Eq.13 is equivalent to the stability under $Gl(k)$ action.

Proof Eq.13⇒ stability. Assume that the point $(\alpha_1, \alpha_2, a, b)$ is unstable then for some g_t

$$(g_t\alpha_1 g_t^{-1}, g_t\alpha_2 g_t^{-1}, a g_t^{-1}, g_t b)$$

stays bounded. Suppose first that $0 < w_1 = \cdots = w_s > w_{s+1} \cdots \geq w_k$, then $(ba)_{1,\cdots s,\cdot} = 0$. Likewise

$$(\alpha_i) = \begin{pmatrix} (M_i)_{s,k-s} & 0 \\ * & * \end{pmatrix}, \quad [\alpha_1, \alpha_2] = \begin{pmatrix} [M_1, M_2] & 0 \\ * & * \end{pmatrix}.$$

We can thus find a common eigenvector v for $M_{1,2}$, $vM_1 = \lambda_1 v$ and $vM_2 = \lambda_2 v$, then writing $v = [v, \underbrace{0, \cdots, 0}_{k-s}]$ $vB = 0$ at the point $-y - z\lambda_2 = x + z\lambda_1 = 0$. This contradicts the surjectivity of B in Eq.13.

Similarly, if $0 > w_1 = \cdots = w_s > w_{s+1} \geq \cdots w_k$, then one must have $a = 0$, and then $[\alpha_1, \alpha_2] = 0$ and one has an eigenvector $\alpha_{1,2} v = \lambda_{1,2} v$, then we found a kernel of A.

Stability⇒Eq.13. Assume first that A has a kernel $\vec{\xi} \in \mathbb{C}^k$, which can be assumed to be $[1, 0_{k-1}]^T$, then

$$\alpha_1 = \begin{bmatrix} -x & \vec{u}^T \\ 0 & * \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} -y & \vec{v}^T \\ 0 & * \end{bmatrix}, \quad a = \begin{bmatrix} 0 & * \end{bmatrix}.$$

Then the orbit of $\alpha_{1,2}, a, b$ under $g_t = \operatorname{diag}[t^{-1}, 1, \cdots, 1]$ remains bounded.

Similarly if B in Eq.13 has a cokernel say $[1, 0_{k-1}]$, then the first row of α_1, α_2, b are $[-x, 0 \cdots, 0]$, $[-y, 0, \cdots, 0]$ and 0. Then their orbit under $g_t = \operatorname{diag}[t, 1, \cdots, 1]$ is bounded, a contradiction. Note that the proof of the reverse direction does not use $[\alpha_1, \alpha_2] + ba = 0$ ■

Summarising we have

Proposition 3.4 *The instanton moduli space is the GIT quotient of the set of matrices of thm.3.1 by the GIT quotient, and hence isomorphic to moduli space of holomorphic $Gl(2, \mathbb{C})$ bundles on $\mathbb{P}^2$. These matrices satisfy $[\alpha_1, \alpha_2] + ba = 0$.*

3.4 Relation to the Hilbert scheme of points on $\mathbb{C}^2$

We now turn to the rank one case and we first show that certain simplification of the ADHM data occurs.

Definition 3.5 *We take $\mathbf{k} = \mathbb{C}$, the Hilbert scheme $\mathbf{k}^{[n]}$ of n points on $\mathbf{k}^2$ is defined as ideals $\mathcal{I}$ of $\mathbf{k}[x, y]$ such that $\dim_{\mathbf{k}} \mathbf{k}[x, y]/\mathcal{I} = n$.*

Proposition 3.6 *The Hilbert scheme can be described in ADHM like data. Let $b \in \hom(\mathbf{k}, \mathbf{k}^n)$ and $\alpha_1, \alpha_2 \in \hom(\mathbf{k}^n, \mathbf{k}^n)$ such that there is no subspace $S \subset \mathbf{k}^n$ such that $\alpha_{1,2}(S) \subset S$ and $\operatorname{img} b \subset S$. And one mods out the action of $Gl(\mathbf{k}, n)$.*

Here one might be puzzled by the absence of a map, this can be explained by the next lemma which shows that stability implies $a = 0$.

Lemma 3.7 *Assuming $[\alpha_1, \alpha_2] = -ba$, define $S = \alpha_{i_1} \cdots \alpha_{i_n} b$, with $i_p = 1, 2$. Then $aS = 0$.*

Proof We use an induction on the length of words. First some special cases. Since $ab = -\mathrm{Tr}[\alpha_1, \alpha_2] = 0$, then $a\alpha_i b = \mathrm{Tr}[\alpha_i ba] = -\mathrm{Tr}[\alpha_i[\alpha_1, \alpha_2]] = 0$, also $a\alpha_1\alpha_2 b = -\mathrm{Tr}[\alpha_1\alpha_2[\alpha_1, \alpha_2]] = -\mathrm{Tr}[[\alpha_1, \alpha_2][\alpha_1, \alpha_2]]/2 = \mathrm{Tr}[baba] = 0$. Now within any word $a\alpha_{i_1} \cdots \alpha_{i_n} b$, we can exchange the α_1, α_2 at the cost of $[\alpha_1, \alpha_2] = -ba$

which breaks the word into two smaller words. Thus we can assume the word is rearranged into $a\alpha_1^p\alpha_2^q b$ with $p, q \geq 2$

$$\begin{aligned} \alpha_1^p\alpha_2^q[\alpha_1,\alpha_2] &= \alpha_1^p\alpha_2^q\alpha_1\alpha_2 - \alpha_1^{p+1}\alpha_2^{q+1} \\ &= \alpha_1^p\alpha_2^{q-1}[\alpha_2,\alpha_1]\alpha_2 + \alpha_1^p\alpha_2^{q-1}\alpha_1\alpha_2^2 - \alpha_1^{p+1}\alpha_2^{q+1} \end{aligned} \tag{14}$$

The first term equals minus the lhs up to smaller length words. For the second term

$$\alpha_1^p\alpha_2^{q-1}\alpha_1\alpha_2^2 = \alpha_1^p\alpha_2^{q-2}[\alpha_2,\alpha_1]\alpha_2^2 + \alpha_1^p\alpha_2^{q-2}\alpha_1\alpha_2^3.$$

The first term is again minus the lhs of Eq.14. Keep going the second term eventually turns into $\alpha_1^{p+1}\alpha_2^{q+1}$ cancelling the last term of rhs of Eq.14 ■

Remark It seems that for the ideal sheafs on $\mathbb{C}^2$, one has dropped the injectivity condition in Eq.13 in the ADHM data thm.3.1, which is relavent for vector bundles (locally free sheafs).

Indeed, we can show that the surjectivity of Eq.13 is equivalent to the stability condition of prop.3.6. Suppose such a subspace S exists which then can be assumed to be $[*_s, 0_{k-s}]^T$, then similar to the proof of prop.3.3, we can narrow down $\alpha_i = [*, *; 0, M_i]$, $b = [*; 0]$. Thus $[\alpha_1, \alpha_2] + ba = [*, *; 0, [M_1, M_2]] + [*, *; 0, 0] = 0$, and $[M_1, M_2] = 0$. Then continuing as before (pick a common eigenvector of $M_{1,2}$ and putting it to be $[1; 0]$), we violate the surjectivity.

Suppose now instead that B has a cokernel, say, $[1; 0]$, then the first rows of $\alpha_{1,2}$ and b are $[-x, 0, \cdots, 0]$, $[-y, 0, \cdots, 0]$ and zero. Then we can pick S to be $[0; *; \cdots; *]$.

If instead A in Eq.13 has a kernel, it need not violate the stability condition. We have for example

$$\alpha_1 = \begin{bmatrix} -x & 1 \\ 0 & -x-1 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} -y & z \\ 0 & -y-z \end{bmatrix}, \quad a = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since $\alpha_1 b = -xb + [1; -1]$ and so no such S exists.

Proof of prop.3.6. We first construct map from the ADHM data above to the Hilbert scheme. We define a map $\phi:\ k[x, y] \to k^n$ by sending $1 \to b(1) \in k^n$, and $x\cdot 1 \to \alpha_1 b(1)$ and $y \cdot 1 \to \alpha_2 b(1)$. This is consistent since $\alpha_{1,2}$ commute. The image of ϕ is generated by $\alpha_1^m\alpha_2^n b(1)$. Thus the image is preserved by $\alpha_{1,2}$ and includes $\mathrm{img}(b)$, and hence must be the entire k^n. Thus the kernel of ϕ is the ideal we seek $k[x, y]/\ker\phi \simeq k^n$.

Now from $\mathcal{I} \subset k[x, y]$, since $\dim_k k[x, y]/\mathcal{I}$, the action of x, y on the quotient are two commuting matrices α_1, α_2. And the image of 1 is defined as $\mathrm{img}\, b$. To show the stability condition, if there had been any subspace S such that $b(1) \subset S$, then $\alpha_1^m\alpha_2^n b(1) = k^n$, and so S is not preserved by $\alpha_{1,2}$.

Proposition 3.8 *The Hilbert scheme is smooth.*

Proof We fist show the smoothness before the quotient. The space is cut out by the equation $f = [\alpha_1, \alpha_2] = 0$. We show that the corank of df is constant. If $\xi \in \hom(k^n, k^n)$ is perpendicular to $\delta[\alpha_1, \alpha_2]$ then

$$\mathrm{Tr}[\xi[\delta\alpha_1, \alpha_2] + \xi[\alpha_1, \delta\alpha_2]] = 0$$

meaning $[\xi, \alpha_{1,2}] = 0$. To prove that the cokernel has constant rank, we construct an isomorphism to k^n. We send ξ to $\xi b(1)k^n$. Now for the inverse map, given $v \in k^n$, we define a matrix ξ such that $\xi(\alpha_1^m\alpha_2^n b(1)) = \alpha_1^m\alpha_2^n v$. Since $\alpha_1^m\alpha_2^n b(1) = k^n$, the action of ξ is defined for all k^n. The action thus defined clearly commute with $\alpha_{1,2}$. The two maps are inverse to each other, thus the corank of df is n.

Then one takes the GIT quotient and finishes the proof ■

We can see more explicitly the relation of the ADHM data to points on k^2. We take $n = 2$, assume first that α_1, α_2 both have two distinct eigenvalues and so they are put to the diagonal form by a $GL(k, n)$ action

$$\alpha_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix},$$

then (λ_1, μ_1), (λ_2, μ_2) are the two points k^2. If instead the eigenvalues are the same

$$\alpha_1 = \begin{bmatrix} \lambda & c \\ 0 & \lambda \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} \mu & d \\ 0 & \mu \end{bmatrix}.$$

First we show that c, d cannot be both zero. If not, then the entire k^n is preserved by α_1, α_2, violating the stability condition. Now we assume $c \neq 0$, then $[*, 0]^T$ is preserved by both $\alpha_{1,2}$, so the image $b(1)$ can be chosen as $[0, 1]^T$. Then if $f(x, y)$ is a polynomial, and it is in the ideal if $f(\alpha_1, \alpha_2)b(1) = 0$

$$f(\begin{bmatrix} \lambda & c \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} \mu & d \\ 0 & \mu \end{bmatrix}) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} f(\lambda, \mu) & nc\lambda^{n-1} + nd\lambda^{n-1} \\ 0 & f(\lambda, \mu) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} (c\partial_\lambda + d\partial_\mu) f(\lambda, \mu) \\ f(\lambda, \mu) \end{bmatrix}.$$

So $f \in \mathcal{I}$ if $f(\lambda, \mu) = 0 = (c, d) \cdot \nabla f(\lambda, \mu) = 0$.

3.5 Sod arguments

Let $X = \mathbb{P}^2$, then $D(X) = D^b(Coh(X)) \simeq \langle \mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1) \rangle$. Let F be a rank r holomorphic vector bundle over $\mathbb{P}^2$ with $c_1 = 0$ and $c_2 = d$, and that $F|_{\mathbb{P}^1}$ trivial. By definition of the s.o.d we have a sequence

$$\begin{array}{ccccccc} 0 = F_3 & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 = F \\ & \nwarrow \swarrow & & \nwarrow \swarrow & & \nwarrow \swarrow & \\ & \langle \mathcal{O}(1) \rangle & & \langle \mathcal{O} \rangle & & \langle \mathcal{O}(-1) \rangle & \end{array},$$

that is the cones $C(F_3, F_2) \in \langle \mathcal{O}(1) \rangle$, $C(F_2, F_1) \in \langle \mathcal{O} \rangle$ and $C(F_1, F_0) \in \langle \mathcal{O}(-1) \rangle$. The term F_2 can be described as follows, since $F_3 = 0$, so $C(F_3, F_2) = F_2$, hence

$$F_2 = \mathcal{O}(1) \otimes V,$$

where V is a graded vector space (recall one is allowed to take direct sum and shifts).

To determine V, we consider the cone

$$F_2 \to F_0 \to C(F_2, F_0) \in \langle \mathcal{O}, \mathcal{O}(-1) \rangle = \langle \mathcal{O}(1) \rangle^\perp.$$

Hence $\mathrm{Ext}^i(\mathcal{O}(1), C(F_2, F_0)) = 0$, and we get the equality

$$\mathrm{Ext}^i(\mathcal{O}(1), F_2) = \mathrm{Ext}^i(\mathcal{O}(1), F).$$

Note that the lhs is $\mathrm{Ext}^i(\mathcal{O}(1), F_2) = \mathrm{Ext}^i(\mathcal{O}(1), \mathcal{O}(1) \otimes V) = V$. To determine the rhs, we notice

$$\mathrm{Ext}^0(\mathcal{O}(1), F) = \mathrm{Ext}^2(\mathcal{O}(1), F) = 0,$$

Then the dimension of $\mathrm{Ext}^1(\mathcal{O}(1), F)$ can be computed from the RR theorem which gives

$$d = \dim \mathrm{Ext}^0 - \mathrm{Ext}^1 + \mathrm{Ext}^2 = \int_{\mathbb{P}^2} ch(F) Td(T\mathbb{P}^2),$$

We conclude therefore

$$V = \mathrm{Ext}^i(\mathcal{O}(1), \mathcal{O}(1) \otimes V) = \mathrm{Ext}^i(\mathcal{O}(1), F) = \mathbb{C}^d, \quad i = 1,$$

that is $V \sim \mathbb{C}^d[-1]$ and

$$F_2 = \mathcal{O}(1)^{\oplus d}[-1].$$

The steps leading to F_1 is simlar, consider the DT

$$F_1 \to F \to \mathcal{O}(-1) \otimes V',$$

and $F_1 \sim C(F_3, F_1) \in \langle \mathcal{O}(1), \mathcal{O} \rangle = {}^{\perp}\mathcal{O}(-1)$. So by applying the $\mathrm{Ext}^i(-, \mathcal{O}(-1))$ functor to the above DT, we get

$$\mathrm{Ext}^i(F, \mathcal{O}(-1)) = \mathrm{Ext}^i(\mathcal{O}(-1) \otimes V', \mathcal{O}(-1)) = V'.$$

The computation of $\mathrm{Ext}^i(F, \mathcal{O}(-1))$ is exactly as before, and we get

$$(V')^i = \mathbb{C}^d,\ i = 1,$$

and zero otherwise, hence

$$F_1 \to F \to \mathcal{O}(-1)^{\oplus d}[1]$$

is a DT. We can rotate this DT and get a new DT

$$\mathcal{O}(-1)^{\oplus d} \to F_1 \to F$$

which shows that F_1 is a v.b. of rank $r + d$.

Now look instead at the DT

$$\begin{array}{l} F_2 \to F_1 \to C(F_2, F_1) \to F_2[1] \\ \qquad\qquad\quad\ \mathcal{O} \otimes V'' \quad \mathcal{O}(1)^{\oplus d}. \end{array}$$

We know that F_1 is a v.b. of rank $r + d$, hence V'' is of dimension $r + 2d$ at degree 0, i.e. $\mathcal{O} \otimes V'' \sim \mathcal{O}^{r+2d}$.

To summarise, we have two relations

$$\mathcal{O}(-1)^{\oplus d} \to F_1 \to F, \quad F_1 \to \mathcal{O}^{r+2d} \to \mathcal{O}(1)^{\oplus d}$$

from which we get a presentation for F

$$\mathcal{O}(-1)^{\oplus d} \to \mathcal{O}^{r+2d} \to \mathcal{O}(1)^{\oplus d}$$

as the homology in the middle.

4 Nekrasov Localisation

Summarising the ADHM construction, we have the data $\alpha_{1,2} \in M_k$ and $a \in M_{lk}, \quad b \in M_{kl}$, satisfying $[\alpha_1, \alpha_2] + ba = 0$, $[\alpha_1, \alpha_1^\dagger] + [\alpha_2, \alpha_2^\dagger] + bb^\dagger - a^\dagger a = 0$, quotient by the $U(k)$ action $\alpha_i \to g\alpha_i g^{-1}$, $a \to ag^{-1}$

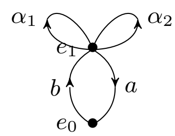


Figure 1: The Nakajima quiver.

and $b \to gb$. Since the volume of $U(k)$ is finite we do not do the quotient but divide in the end by the its volume.

The data is organised as the Nakajima quiver fig.1 which has a hyper-Kähler structure

$$\begin{aligned} J_1(\alpha_1,\alpha_2,a,b) &= (i\alpha_1, i\alpha_2, ia, ib), \\ J_2(\alpha_1,\alpha_2,a,b) &= (-\alpha_2^\dagger, \alpha_1^\dagger, -b^\dagger, a^\dagger), \\ J_3(\alpha_1,\alpha_2,a,b) &= (-i\alpha_2^\dagger, i\alpha_1^\dagger, -ib^\dagger, ia^\dagger). \end{aligned}$$

We will use the following $U(1)$'s to compute the volume of the moduli space. The rotation of phases of α_1, α_2, a, b

$$\begin{aligned} &(\alpha_1,\alpha_2,a,b) \to (e^{i\epsilon_1}\alpha_1, \alpha_2, a, e^{i\epsilon_1}b) \\ &(\alpha_1,\alpha_2,a,b) \to (\alpha_1, e^{i\epsilon_2}\alpha_2, a, e^{i\epsilon_2}b) \\ &(\alpha_1,\alpha_2,a,b) \to (g\alpha_1 g^{-1}, g\alpha_2 g^{-1}, ag^{-1}, gb), \quad g = e^{i\phi},\ \phi \in \mathfrak{t} \subset \mathfrak{u}(k) \\ &(\alpha_1,\alpha_2,a,b) \to (\alpha_1, \alpha_2, ga, bg^{-1}), \quad g = e^{ix},\ x \in \mathfrak{t} \subset \mathfrak{u}(\ell). \end{aligned}$$

If one looks at how to pass from $\alpha_{1,2}$ to points on $\mathbb{C}^2$, the first two rotations above correspond to the rotation of the two coordinates of $\mathbb{C}^2$. The third and fourth one are the Cartan of $U(k)$ and $U(\ell)$.

Combining the complex structures with the Hermitian form $\langle -,- \rangle = ||\alpha_1||^2 + ||\alpha_2||^2 + ||a||^2 + ||b||^2$, we get three symplectic forms

$$\begin{aligned} \omega_1 &= \langle J_1 -, - \rangle = i\mathrm{Tr}[d\alpha_1 d\bar\alpha_1 + d\alpha_2 d\bar\alpha_2] + i\mathrm{Tr}[dad\bar a] + i\mathrm{Tr}[dbd\bar b], \\ \omega_2 &= \langle J_2 -, - \rangle = 2\mathrm{Re}\,\mathrm{Tr}[d\alpha_1 d\alpha_2] + 2\mathrm{Re}\,\mathrm{Tr}[dadb], \\ \omega_3 &= \langle J_3 -, - \rangle = 2\mathrm{Im}\,\mathrm{Tr}[d\alpha_1 d\alpha_2] + 2\mathrm{Im}\,\mathrm{Tr}[dadb]. \end{aligned}$$

From the hyperKähler structure, the ADHM moduli space can be viewed as a hyperKähler reduction of the linear space $\mathbb{V} = \{\alpha_{1,2} \in M_k,\ a \in M_{\ell,k},\ b \in M_{k,\ell}\}$ by $U(k)$

$$\begin{aligned} &\mathcal{M} = \mu_c^{-1}(0) \cap \mu_r^{-1}(0)/G, \\ &\mu_c \sim [\alpha_1,\alpha_2] - ba, \quad \mu_r \sim [\alpha_1,\alpha_1^\dagger] + [\alpha_2,\alpha_2^\dagger] - a^\dagger a + b^\dagger b. \end{aligned}$$

Check the signs.

To find the fixed points of the torus actions, we start from b

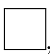
$$b \to e^{i\epsilon_1 + i\epsilon_2} b + gb + bh^{-1}, \quad g \in T(k), \quad h \in T(\ell).$$

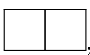
For this to be satisfied one must have

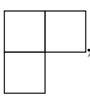
$$b = \begin{pmatrix} * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

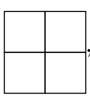
This conveys the idea that the $U(\ell)$ instanton breaks up into a number of $U(1)$ instantons. Now we focus on each column, and assume that $b \in M_{k,1}$. Then from the earlier discussion we know $a = 0$ and the ideal $\mathcal{I}$ is defined such that $f \in \mathcal{I}$ if $f(\alpha_1, \alpha_2)b = 0$. The solution correspond to Young-diagrams.

Example

 $, \quad \alpha_1 = \alpha_2 = 0, \quad b = \sqrt{\zeta},$

 $, \quad \alpha_1 = \begin{pmatrix} 0 & 0 \\ \sqrt{\zeta} & 0 \end{pmatrix}, \quad \alpha_2 = 0, \quad b = \begin{pmatrix} \sqrt{2\zeta} \\ 0 \end{pmatrix},$

 $, \quad \alpha_1 = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{\zeta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{\zeta} & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} \sqrt{3\zeta} \\ 0 \\ 0 \end{pmatrix},$

 $, \quad \alpha_1 = \sqrt{\zeta} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \end{pmatrix}, \quad \alpha_2 = \sqrt{\zeta} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 2\sqrt{\zeta} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$

From this point on, we need to compute $e(\mathrm{Ext}(\mathcal{I}_\pi, \mathcal{I}_\pi))$ as the Euler class of the normal bundle of the fixed point. The computation has been done already in the DT theory notes.

4.1 Nekrasov's approach

The equivariant complex can be written as

$$\begin{aligned}
&\delta\alpha_i = \psi_{\alpha_i}, \quad \delta\psi_{\alpha_i} = [\phi, \alpha_i] + \epsilon_i \alpha_i, \\
&\delta a = \psi_a, \quad \delta\psi_a = -a\phi + xa, \\
&\delta b = \psi_b, \quad \delta\psi_b = \phi b - bx + (\epsilon_1 + \epsilon_2) b, \\
&\delta\chi_r = H_r, \quad \delta H_r = [\phi, \chi_r], \\
&\delta\chi_c = H_c, \quad \delta H_c = [\phi, \chi_c] + (\epsilon_1 + \epsilon_2)\chi_c.
\end{aligned}$$

Here $H_{r,c}$ are introduced to realise the two constraints Eq.12 cohomologically.

We define the volume by the localisation formula

$$\int \mathrm{Vol} \equiv \sum_{f.p.} \frac{1}{e}.$$

The fixed points are characterised as

$$0 = (\phi_I - \phi_J + \epsilon_i)(\alpha_i)_{IJ},$$
$$0 = (x_p - \phi_I)a_{pI},$$
$$0 = (\phi_I - x_p + \epsilon_1 + \epsilon_2)b_{Ip}, \quad I = 1, \cdots, k; \quad p = 1, \cdots, \ell. \tag{15}$$

To compute the Euler character at the fixed points, we compute the Chern character

$$\chi = (e^{\epsilon_1} + e^{\epsilon_2} - e^{\epsilon_1+\epsilon_2} - 1)\sum_{IJ} e^{\phi_{IJ}} + \sum_{p,I}\big(e^{-\phi_I+x_p} + e^{\phi_I - x_p+\epsilon_1+\epsilon_2}\big).$$

The Euler character is the product of the weights

$$e^{-1} = \frac{(\epsilon_1+\epsilon_2)}{\epsilon_1\epsilon_2}\prod_{I\neq J}\frac{\big(\phi_{IJ}+\epsilon_1+\epsilon_2\big)\phi_{IJ}}{\big(\phi_{IJ}+\epsilon_1\big)\big(\phi_{IJ}+\epsilon_2\big)}\prod_{p,I}\frac{1}{\big(-\phi_I+x_p\big)\big(\phi_I - x_p+\epsilon_1+\epsilon_2\big)}.$$

We have to integrate over the Cartan of $U(k)$, i.e. over ϕ_I

$$\int\prod_I d\phi_I\prod_{I\neq J}\frac{\big(\phi_{IJ}+\epsilon_1+\epsilon_2\big)\phi_{IJ}}{\big(\phi_{IJ}+\epsilon_1\big)\big(\phi_{IJ}+\epsilon_2\big)}\prod_{p,I}\frac{1}{\big(-\phi_I+x_p\big)\big(\phi_I - x_p+\epsilon_1+\epsilon_2\big)}$$

The fixed points Eq.15 correspond to the possible residues of the integral above. Out of the equations of Eq.15, we need to pick k of them and solve for ϕ_I. But we have to use $\phi_I - x_p = 0$ at least once, since the other equation only depend on $\phi_I - \phi_J$. We assume

$$\phi_1 = x_p,$$

then for ϕ_2, we cannot use $\phi_2 - x_p$ anymore since the numerator vanishes if $\phi_1 = \phi_2$. So we use say $\phi_2 = \phi_1 + \epsilon_1$, Nor can we use again $\phi_2 = \phi_1 \pm (\epsilon_1 + \epsilon_2)$ since the numerator vanishes. For ϕ_3, we can

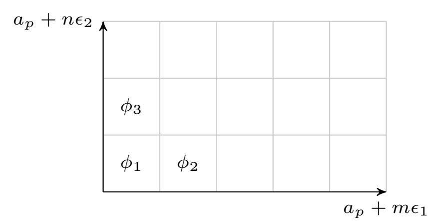


Figure 2: The arrangement of poles.

choose $\phi_3 = \phi_2 + \epsilon_1$, or $\phi_3 = \phi_1 + \epsilon_2$, but not $\phi_3 = \phi_2 + \epsilon_2$, since this will again cause $\phi_3 - \phi_1 = \epsilon_1 + \epsilon_2$ to vanish. We fix $\phi_3 = \phi_1 + \epsilon_2$, in the next step we can pick $\phi_4 = \phi_2 + \epsilon_1$, this is because we have two poles $\phi_4 - \phi_2 = \epsilon_2$, $\phi_4 - \phi_3 = \epsilon_1$ and one zero $\phi_4 - \phi_1 = \epsilon_1 + \epsilon_2$. To summarise, the arrangements of ϕ_I correspond to Young-diagrams. In general, we will use $\phi_I = a_p$ a number of times, and this gives the partition of $\phi_I, I = 1\cdots, k$ into ℓ Young-diagrams.