Contents

1 Analogy with Chern-Simons

Recall the Chern-Simons functional

$$CS(A) = \frac{k}{4\pi} \operatorname{Tr} \int_{M^3} AdA + \frac{2}{3} A^3 : \quad B(A) \to \mathbb{R}/\mathbb{Z}$$

regarded as a function from $B(A)$ the space of all connections of a fixed bundle to $\mathbb{R}/\mathbb{Z}$. The critical points of this functional

$$0 = \delta CS(A) = \frac{k}{4\pi} \operatorname{Tr} \int_{M^3} 2\delta A \wedge F$$

leads to flat connections. The *Casson* invariant of M is the Euler class of $B(A)$, and it can be computed by counting the critical points of $CS(A)$ with the appropriate sign (Morse-index).

There is a clear complex analogy to the CS theory. Let X be a CY 3-fold and ρ be the nowhere vanishing hollomorphic 3-form ϱ. Consider the functional

$$CS_{\mathbb{C}}(A) = \frac{k}{?4\pi?} \operatorname{Tr} \int_X (\bar{A}\bar{\partial}\bar{A} + \frac{2}{3}\bar{A}^3)\varrho$$

where $\bar{A}$ is the (0,1) component of the complex connection A. The critical points

$$0 = \delta CS_{\mathbb{C}}(A) \sim \operatorname{Tr} \int_X 2\delta\bar{A} \wedge F^{0,2} \wedge \varrho$$

corresponds to $F^{0,2} = 0$, i.e. holomorphic vector bundles.

In the complex setting, one needs to take the quotient w.r.t. $\mathcal{G}_{\mathbb{C}}$ the complex gauge group, and geometrical invariant theory tells us that one should delete some unstable orbits and then take the quotient (certain bad orbits may be kept for functorial reasons). In the current setting, this means we should consider connections of stable bundles. Recall that a holomorphic vector bundle E is stable if for every $F \subsetneq E$ subbundle

$$\frac{\deg_\omega F}{\operatorname{rk} F} < \frac{\deg_\omega E}{\operatorname{rk} E}.$$

A consequence of the stability is that

$$H^0(X, \operatorname{End} E) = \mathbb{C}.$$

If one is willing to leave the purely algebraic setting and deal with symplectic geometry, then one can take the quotient by means of symplectic reduction, which provides a pathway to physics. Consider the symplectic form

$$\Omega = \operatorname{Tr} \int_X \omega^2 \wedge \delta A \wedge \delta\bar{A}.$$

The gauge transformation $A \to A + d_A \epsilon$ has moment map

$$\langle \mu, \epsilon \rangle = \mathrm{Tr} \int_X \omega^2 \wedge F\epsilon \sim \mathrm{Tr} \int_X (\omega * F)\epsilon.$$

Let κ be a constant then κI is a central element in the gauge group, so we can take the quotient of the level surface $\mu^{-1}(\kappa I)$ w.r.t the real gauge group and get

$$\omega * F = \kappa I, \quad F^{0,2} = 0.$$

More generally we can consider the Hermitian Yang-Mills equations

$$F^{0,2} = 0, \quad \omega * F = \kappa I,$$

where κ is any function, but it can always be set to a constant, by shifting the corresponding Hermitian metric. The constant κ is related to the slope as

$$\deg_\omega E = \int c_1 \omega^2 = \mathrm{Tr} \int F\omega^2 \sim \mathrm{Tr} \int \omega * F = \mathrm{Tr} \int \kappa I = \mathrm{rk}\, E\, \mathrm{Vol}_X.$$

Remark From the HYM equations one can derive some vanishing theorems. Let s be a holomorphic section of E, then

$$0 = |\bar\partial s|^2 = -\langle s^*, \omega^{j\bar i} \nabla_j \bar\partial_{\bar i} s \rangle = -\langle s^*, \omega^{j\bar i} F_{j\bar i} s \rangle - \langle s^*, \omega^{j\bar i} \bar\partial_{\bar i} \nabla_j s \rangle = -\kappa \langle s^*, s \rangle + |\nabla s|^2.$$

So if $\kappa < 0$ we have $s = 0$ and if $\kappa = 0$ then s is covariantly constant.

2 Virtual deformation theory

Since we would like to count stable bundles, it is natural to extend the counting to include also the stable coherent sheaves as a compactification of the moduli space. We hold fixed the topological type of E and deform its holomorphic structure, i.e. the $\bar\partial$ operator. The infinitesimal deformation of the connection is controlled by

$$H^{0,1}(X, \mathrm{End}\, E)_0,$$

where the subscript $_0$ denotes the trace free part, since we are also holding fixed the determinant line bundle $\det E$. We cannot consider this as a spanning a vector bundle over the moduli space of stable bundles even when the latter is smooth, this is because the dimension of the group has jumps. One should instead consider a more invariant complex of bundles

$$H^{0,1}(X, \mathrm{End}\, E)_0 \oplus H^{0,2}(X, \mathrm{End}\, E)_0 \oplus H^{0,3}(X, \mathrm{End}\, E)_0,$$

where $H^{0,0}(X, \mathrm{End}\, E)_0$ is missing because $H^{0,0}(X, \mathrm{End}\, E) = \mathbb{C}$. Since the signed dimension of this complex is given by the Hirzbruch genus which is topological. One can consider the counting problem on X not necessarily CY, but if $K_X = 0$ then Serre duality

$$H^{0,i}(X, \mathrm{End}\, E) \simeq H^{0,3-i}(X, \mathrm{End}\, E^*) \simeq H^{0,3-i}(X, \mathrm{End}\, E)$$

shows that the virtual dimension vanishes, and so we indeed have a counting problem. If we extend E to coherent sheafs $\mathcal{E}$, then the deformation complex should be modified accordingly

$$\mathrm{Ext}^1(\mathcal{E},\mathcal{E})_0 \oplus \mathrm{Ext}^2(\mathcal{E},\mathcal{E})_0 \oplus \mathrm{Ext}^3(\mathcal{E},\mathcal{E})_0.$$

However from concerns of virtual deformation theory, we would like to have a two term presentation of the tangent complex, so we would like $\mathrm{Ext}^3(\mathcal{E},\mathcal{E})_0 = 0$. Assuming this, the virtual dimension of the moduli space is $\dim \mathrm{Ext}^1(\mathcal{E},\mathcal{E})_0 - \dim \mathrm{Ext}^2(\mathcal{E},\mathcal{E})_0$, which presumably has good invariance property, being related to the Hirzbruch genus. Let $[\mathcal{M}]$ denote the fundamental class of the moduli space, and ideally the DT invariants are

$$\int_{[\mathcal{M}]} \mathcal{O},$$

where $\mathcal{O}$ is some observable, i.e. the DT invariants are just some intersection theory on $\mathcal{M}$. But sometimes the dimension of the moduli space is larger than the virtual dimension, and what is worse does not evince the same invariance property as the virtual dimension does.

Example Take the Gromov-Witten theory, see `http://www.math.utah.edu/~yplee/research/grenoble.pdf`. Let

$$F_{GW} = \sum_{g\geq 0,\beta} F_\beta^g u^{2g-2} t^\beta,$$

where the sum β is over $H_2(X)$. Consider now F_0^g, i.e. the image of Σ_g is a point. In this case the moduli space is simple

$$\mathcal{M}_{g,0} \times X$$

of dimension $\dim \mathcal{M}_{g,0} + \dim X$ (where $\dim \mathcal{M}_{g,n} = 3g - 3 + n$), but the virtual dimension is $\dim \mathcal{M}_{g,0} + \dim H^0(\Sigma_g, f^*TX) - \dim H^1(\Sigma_g, f^*TX)$ and

$$\dim H^0(\Sigma_g, f^*TX) - \dim H^1(\Sigma_g, f^*TX) = (\dim H^0(\Sigma_g) - \dim H^1(\Sigma_g)) \dim X = (1-g)\dim X.$$

The mismatch of dimension shows that the virtual fundamental class cannot be the same as the actual fundamental class.

Thomas showed that one can construct a virtual fundamental class (he requires the 'perfect obstruction theory') in the case of *ideal sheafs*

Definition An ideal sheaf is a rank 1 torsion free sheaf of trivial determinant.

Remark There is a canonical map $\mathcal{I} \to \mathcal{I}^{\vee\vee}$, whose kernel is the torsion part of $\mathcal{I}$, thus we have an embedding

$$0 \to \mathcal{I} \to \mathcal{I}^{\vee\vee}.$$

The double dual $\mathcal{I}^{\vee\vee}$ is a reflexive sheaf, and from a result of Hartshorne, rank one reflexive sheafs are locally free, i.e. line bundles. The determinant of I is defined as the determinant of $\mathcal{I}^{\vee\vee}$, hence $\mathcal{I}^{\vee\vee} \simeq \mathcal{O}_X$. So $\mathcal{I}$ is a subsheaf of $\mathcal{O}$, this is the reason it is called the ideal sheaf. An ideal sheaf determines an subscheme

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0. \tag{1}$$

Due to the constraint on the determinant the dimension of Y is at most 1. This is the rough reason why DT theory is related to the GW theory. One can also construct higher rank DT theories, but their relation to GW theory is unclear.

We consider Y of fixed Euler number $n = \chi(Y)$, and homology class $[Y] = \beta \in H_2(X, \mathbb{Z})$, and the moduli space $I_n(X, \beta)$ is the Hilbert scheme of curves of X. We denote by $[I_n(X, \beta)]^{vir}$ the virtual fundamental class of the moduli space. Its virtual dimension is zero and we define

$$N_{n,\beta} = \int_{[I_n(X,\beta)]^{vir}} 1 \tag{2}$$

as the counting of such schemes.

3 GW/DT at degree 0

The degree 0 GW free energy is given by

$$F_{0,g\geq 2} = (-1)^g u^{2g-2} \underbrace{\frac{1}{2}\int_X (c_3(X) - c_1(X)c_2(X))}_{\#} \int_{\bar{M}_g} \lambda_{g-1}^3,$$

where λ_{g-1} denotes $(g-1)^{th}$ Chern class of the Hodge bundle $\mathbb{E}$ over $\bar{M}_g$, whose fibre is generated by $H^0(X, K_X) \simeq H^1(X)^\vee$. The virtual class is the top Chern class of the bundle $TX \otimes \mathbb{E}^\vee$, which can be computed using the spliting principle. Letting $c(TX) = (1+x_1)(1+x_2)(1+x_3)$

$$c_{top}(TX \otimes \mathbb{E}^\vee) = (-1)^g(\lambda_g - x_1\lambda_{g-1} + x_1^2\lambda_{g-2} - x_1^3\lambda_{g-3})(x_1 \to x_2)(x_2 \to x_3)$$

and use the formula $\lambda_g^2 = 0$, $\lambda_{g-1}^2 = 2\lambda_g\lambda_{g-2} = 0$, one gets

$$\frac{1}{2}(c_3 - c_2c_1)\lambda_{g-1}^3.$$

The Hodge integral is computed to be

$$\int_{\bar{M}_g} \lambda_{g-1}^3 = \frac{|B_{2g}|}{2g}\frac{|B_{2g-2}|}{2g-2}\frac{1}{(2g-2)!}.$$

It is claimed that if one performs the sum over g, one obtains

$$Z_{GW} = \exp F = M(e^{iu})^{\#}.$$

As for the DT theory at degree 0, we consider first the simplest case $N_{1,0}$. The moduli space is clearly the entire X, here we have an instance where the virtual class and the actual moduli space are of different dimension. In this case the virtual fundament class is just the top Chern class of the obstruction bundle $\mathrm{Ext}^2(\mathcal{I}, \mathcal{I})$. To compute this group we can assume that X is affine, since $\mathcal{I}$ represents a sheaf that is supported at a point. Let us compute $\mathrm{Ext}^i(\mathcal{I}, \mathcal{I})$, from the short exact sequence Eq.1, we get

$$\mathrm{Ext}^i(\mathcal{I}, \mathcal{I}) \simeq \mathrm{Ext}^{i-1}(\mathcal{I}, \mathcal{O}_Y) \simeq \mathrm{Ext}^i(\mathcal{O}_Y, \mathcal{O}_Y).$$

By definition we need to find a resolution of $\mathcal{O}_Y$, which we take to be the Koszul resilution, let $Y = \{p\} \in X$, then

$$\cdots \to \wedge^2 T^*X \to \wedge^1 T^*X \to \wedge^0 T^*X \to \mathcal{O}_Y \to 0,$$

with the differential defined by contracting with a section s of TX vanishing transversely at p. Thus

$$\underline{\mathrm{Hom}}(\wedge^i T^*X, \mathcal{O}_Y) = \wedge^i TX|_p,$$

thus

$$\mathrm{Ext}^i(\mathcal{O}_Y, \mathcal{O}_Y) = \wedge^i TX|_p.$$

In particular the deformation $\mathrm{Ext}^1(\mathcal{I}, \mathcal{I}) = TX|_p$ as expected (this is an instance of unobstructed deformation). The top Chern class of the obstruction bundle is

$$c_3(\wedge^2 TX) = c_3(K^* \otimes T^*X) = -c_3(K \otimes TX) = -(c_3(X) - c_1(X)c_2(X)).$$

Integrating this over the moduli space (which is just X) we get

$$N_{1,0} = \int_X -c_3(K \otimes TX) = -(c_3(X) - c_1(X)c_2(X)).$$

The general structure is conjectured to be

$$Z_{DT}(X, q)_0 = M(-q)^{2\#}. \tag{3}$$

4 Toric case

Now we have a torus $\mathbf{T} = T^3$ action, we can localize the integral Eq.2 to the fixed points, more specifically the virtue class

$$[I_n(X, \beta)]^{vir} = \sum_{f.p} \frac{[I_n^T(X, \beta)]^{vir}}{e(N^{vir})},$$

where I have used the same symbol for the fundamental class and its dual and N^{vir} is the virtual normal bundle at a fixed point.

So the computation centres on the T fixed ideal sheafs, which have a nice description in a toric manifold. As an example, consider $\mathbb{C}^2$ whose toric fan is just the first quadrant. Each lattice point (i, j) corresponds to a monomial $x^i y^j$. To a plane partition π of fig.1, one associates the ideal

$$\mathcal{I}_\pi = \{x^i y^j \mid (i, j) \notin \pi\}.$$

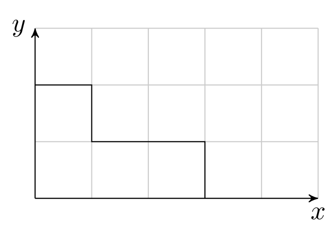

Figure 1: Plane partition π, the ideal I_π is generated by (x^3, xy, y^2).

In general, a T invariant ideal corresponds to placing boxes on the toric fan. Now it is more convenient to sum over $\beta \in H_2(Z,\mathbb{Z})$, for example, the 2D partitions along the edges will lead to non-trivial 2-cycles.

• general strategy of the computation of the Euler characteristic. We first compute the equivariant character of $\mathrm{Ext}^1(\mathcal{I},\mathcal{I}) - \mathrm{Ext}^2(\mathcal{I},\mathcal{I})$, for example, for $\mathbb{C}^3$ and $\mathcal{I}$ corresponding to just one box, one has (see example 4)

$$\mathrm{Ext}^1(\mathcal{I},\mathcal{I}) = t_1^{-1} + t_2^{-1} + t_3^{-1}, \quad \mathrm{Ext}^2(\mathcal{I},\mathcal{I}) = t_1^{-1}t_2^{-1} + t_2^{-1}t_3^{-1} + t_3^{-1}t_1^{-1}, \tag{4}$$

one can now pick an arbitrary squashing parameter $\omega_{1,2,3}$ and then the weights of the above are ω_1, ω_2, ω_3 and $-\omega_1$, $-\omega_2$, $-\omega_3$, and thus the product of weights are

$$\frac{(-\omega_1)(-\omega_2)(-\omega_3)}{(\omega_1+\omega_2)(\omega_2+\omega_3)(\omega_3+\omega_1)}. \tag{5}$$

which will be the weighting of the ideal of one box.

More generally, it is convenient to compute

$$-\chi(\mathcal{I},\mathcal{I}) + \chi(\mathcal{O}_X,\mathcal{O}_X), \tag{6}$$

where $\chi(\mathcal{O}_X,\mathcal{O}_X) = \mathrm{Ext}^0(\mathcal{O}_X,\mathcal{O}_X) = \mathbb{C}$ for a compact space and it serves to subtract $\mathrm{Ext}^0(\mathcal{I},\mathcal{I}) = \mathbb{C}$. But as the toric manifold are in likely not compact, we $\chi(\mathcal{O}_X,\mathcal{O}_X) \neq \mathbb{C}$, but we still take Eq.6 as the definition of the equivariant character.

To advantage of computing the character $\chi(\mathcal{I},\mathcal{I})$ is that one needs not compute any cohomology. We pick a Čech cover of X according to the toric diagram, i.e. $U_\alpha \simeq \mathbb{C}^3$ for each 3-valent vertex and $U_{\alpha\alpha} \simeq \mathbb{C}^2 \times \mathbb{C}^\times$ for each edge connecting two vertices α, β. And $U_{\alpha\beta\gamma}$ for triple intersections.

Now apply the Čech resolution and the character is given by the double complex

$$\sum_{i=0}^{3}(-1)^i\,\mathrm{Ext}^i(\mathcal{I},\mathcal{I}) = \sum_{i,j=0}^{3}(-1)^{i+j}\check{H}^j(X,\mathrm{Ext}^i(\mathcal{I},\mathcal{I})).$$

In fact we do not even need to compute the Cech cohomology

$$\chi(\mathcal{I},\mathcal{I}) = \sum_{i,j=0}^{3}(-1)^{i+j}\check{C}^j(X,\mathrm{Ext}^i(\mathcal{I},\mathcal{I})).$$

Thus we need only compute the ext groups over patches and their intersections, which is the reasoning behind the cutting and pasting in the current setting.

Remark We only need to consider U_α and $U_{\alpha\beta}$ because over triple intersections, the ideal is necessarily empty since otherwise $\det\mathcal{I}$ will not be trivial.

We compute on $\mathbb{C}^3$ first, we claim first that the Euler character will be $(-1)^{\chi(Y)}$ regardless of round which ideal, hence the answer will simply be

$$\sum_{n>0} N_{n,0}q^n = \sum_{n>0}(-1)^n q^n(\#\text{ of 3D partitions}) = M(-q).$$

• The resolution. We need to resolve the ideal $\mathcal{I}_\pi$

$$\cdots \to F_1 \to F_0 \to \mathcal{I} \to 0.$$

Pick generators $\{m_1, \cdots, m_s\}$ of $\mathcal{I}_\pi$, where m_i are monomials. For each subset J of $\{1, \cdots, s\}$, we denote by

$$m_J = \text{lcm}(m_i,\ i \in J).$$

We set F_0 as the free $R = \mathbb{C}[x_1, x_2, x_3]$ module $\oplus R m_i$, with the obvious map to $\mathcal{I}$. For $F_k,\ k \geq 1$, pick subsets J of $I = \{1, \cdots, s\}$ of cardinality $k+1$, and let

$$F_k = \bigoplus_{J \subset I, |J| = k+1} R m_J,$$
$$\partial m_J = \frac{m_J}{m_{J \backslash i_0}} m_{J \backslash i_0} - \frac{m_J}{m_{J \backslash i_1}} m_{J \backslash i_1} \pm \cdots, \quad J = \{i_0, \cdots, i_k\}$$

where the fractions are regarded as elements of the ring and $m_{J \backslash i_j}$ are regarded as the generators. Extending ∂ above linearly, one gets the differential $\partial : \ F_k \to F_{k-1}$.

Example As an example, we just take π to be one box, then the generators can be chosen as $\{x_1, x_2, x_3\}$, and the resolution is similar to the Koszul resolution, with a shift. To be more explicit ξ_i be of degree 1, then

$$F_k = \mathbb{C}[x_i, \xi_i]_{k+1}, \quad \partial = \sum x_i \partial_{\xi_i}.$$

Or when written out

$$0 \to R\cdot x_1 x_2 x_3 \stackrel{\partial_2}{\to} R\cdot x_1 x_2 \oplus R\cdot x_2 x_3 \oplus R\cdot x_2 x_3 \stackrel{\partial_1}{\to} R\cdot x_1 \oplus R\cdot x_2 \oplus R\cdot x_3 \stackrel{\epsilon}{\to} \mathcal{I} \to 0.$$

$$\epsilon = (x_1, x_2, x_3), \quad \partial_1 = \left(\begin{array}{ccc} -x_2 & 0 & x_3 \\ x_1 & -x_3 & 0 \\ 0 & x_2 & -x_1 \end{array} \right), \quad \partial_2 = \left(\begin{array}{c} x_3 \\ x_1 \\ x_2 \end{array} \right).$$

For this simple case one can compute the cohomology of

$$0 \to \hom(F_0, \mathcal{I}) \stackrel{\partial_1^*}{\to} \hom(F_1, \mathcal{I}) \stackrel{\partial_2^*}{\to} \hom(F_2, \mathcal{I}) \to 0,$$

the kernel and cokernels are

$$\begin{aligned} &\ker \partial_1^* = R(x_1, x_2, x_3), \\ &\ker \partial_2^* = R(-x_1, x_3, 0) \oplus R(-x_2, 0, x_3) \oplus R(0, x_2, x_1), \quad \text{img}\, \partial_1^* = (\mathcal{I}, \mathcal{I}, \mathcal{I}) \partial_1, \\ &\text{img}\, \partial_2^* = (\mathcal{I}, \mathcal{I}, \mathcal{I}) \partial_2, \end{aligned}$$

so the ext groups are

$$H_0 = \frac{1}{(1-t_1)(1-t_2)(1-t_3)}, \quad H_1 = t_1^{-1} + t_2^{-1} + t_3^{-1}, \quad H_2 = t_1^{-1} t_2^{-1} \oplus t_2^{-1} t_3^{-1} \oplus t_3^{-1} t_1^{-1}.$$

From this we get that the Euler character of the virtual tangent bundle is (-1) as in Eq.5

Now we do the computation in general, given the resolution $F_\bullet \to I$, one still needs to work out $\hom(F_i, \mathcal{I})$ as in the example above. But again since we are interested in the character, we resolve also the second $\mathcal{I}$ and

$$\sum_i (-1)^i \hom(F_i, \mathcal{I}) = \sum_i (-1)^{i+j} \hom(F_i, F_j).$$

Since F_i are free, the maps are determined on the generators

$$\hom(F_i, F_j) = R(m_J \to m_{J'}),$$

where m_J, $m_{J'}$ are generators of F_i and F_j respectively. Thus the character of $\hom(F_i, F_j)$ is

$$\frac{1}{(1-t_1)(1-t_2)(1-t_3)} \frac{ch(m_{J'})}{ch(m_J)}.$$

Doing the summation over the generators

$$\chi(\mathcal{I}, \mathcal{I}) = \sum (-1)^{i+j} ch(\hom(F_i, F_j)) = \frac{1}{(1-t_1)(1-t_2)(1-t_3)} P(t)P(t^{-1}),$$

where

$$P(t) = \sum (-1)^i \sum_{J,|J|=i+1} ch(m_J)$$

is the Poincare polynomial of the complex $F_\bullet$. One may rewrite χ as

$$\chi(\mathcal{I}, \mathcal{I}) = \frac{1}{(1-t_1)(1-t_2)(1-t_3)} \Big((1-P(t))(1-P(t^{-1})) + P(t) - 1 + P(t^{-1}) - 1 + 1 \Big),$$

the last 1 term will be cancelled by $\chi(\mathcal{O}_X, \mathcal{O}_X)$, indeed, for $\mathcal{O}_X$ there is no need for resolution and the character is $\prod_{i=1,2,3}(1-t_i)^{-1}$. The constituent

$$Q(t) = \frac{1-P(t)}{(1-t_1)(1-t_2)(1-t_3)}$$

has a simple interpretation, it is

$$ch(\mathcal{O}_Y) = ch(\mathcal{O}_X/\mathcal{I}) = ch(\mathcal{O}_X) - ch(\mathcal{I}) = \frac{1}{(1-t_1)(1-t_2)(1-t_3)} - \frac{P(t)}{(1-t_1)(1-t_2)(1-t_3)},$$

which is hence always a *finite* polynomial if the partition is finite. Now χ is written as

$$\chi_\alpha = \chi(\mathcal{O}, \mathcal{O}) - \chi(\mathcal{I}, \mathcal{I}) = Q_\alpha(t) - \frac{Q_\alpha(t^{-1})}{t_1 t_2 t_3} + Q_\alpha(t) Q_\alpha(t^{-1}) \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1 t_2 t_3}$$

we have used the subscript α to denote that this is the vertex contribution.

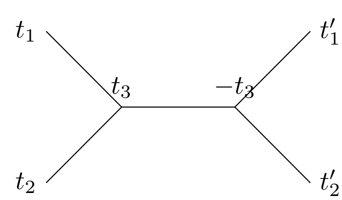

Figure 2: Assume $t_1 + a t_3 = t_1'$ and $t_2 + b t_3 = t_2'$, $a, b \in \mathbb{Z}_{\geq 0}$, the local geometry is that of $\mathcal{O}(-a) \oplus \mathcal{O}(-b) \to \mathbb{P}^2$

For the edges, the computation is similar, except now the ring becomes $R[x_3^{-1}] = \mathbb{C}[x_1, x_2, x_3, x_3^{-1}]$. Thus the restriction of I to the edge is now generated by monomials of x_1, x_2. The rest of the computation remains the same, except that the character of $R[x_3^{-1}]$ is

$$\frac{1}{(1-t_1)(1-t_2)} \delta(1-t_3),$$

where $\delta(1-t)=\sum_{-\infty}^{\infty} t^i$. The character reads now

$$\chi_{\alpha\beta}=\delta(1-t_3)\Big(-Q_{\alpha\beta}(t)-\frac{Q_{\alpha\beta}(t^{-1})}{t_1t_2}+Q_{\alpha\beta}(t)Q_\alpha(t^{-1})\frac{(1-t_1)(1-t_2)}{t_1t_2}\Big),$$

where the over minus sign compare to the vertex term is due to the $(-1)^j$ from the Čech part. The final result is simply the sum of the vertex and edge contributions. But so far, both are infinite polynomials in t_3, we need to reshuffle the terms so that both become finite.

We break $\delta(1-t_3)$ as

$$\frac{1}{1-t_3}+\frac{t_3^{-1}}{1-t_3^{-1}},$$

where by $1/(1-t)$ we mean $1+t+t^2+\cdots$. We combine

$$\begin{aligned}&Q_\alpha(t)-\frac{Q_\alpha(t^{-1})}{t_1t_2t_3}+Q_\alpha(t)Q_\alpha(t^{-1})\frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3}+\\ &\frac{1}{1-t_3}\Big(-Q_{\alpha\beta}(t)-\frac{Q_{\alpha\beta}(t^{-1})}{t_1t_2}+Q_{\alpha\beta}(t)Q_\alpha(t^{-1})\frac{(1-t_1)(1-t_2)}{t_1t_2}\Big),\end{aligned} \tag{7}$$

since $Q_\alpha-Q_{\alpha\beta}$ is finite at $t_3=1$, these are finite polynomials.

However from the change of variables as one passes from α to β patch, $t_1'=t_1t_3^a,\ t_2'=t_2t_3^b$, there will be a finite remainder from the edge contribution

$$\frac{t_3^{-1}}{1-t_3^{-1}}F_{\alpha\beta}(t_1,t_2)-\frac{1}{1-t_3^{-1}}F_{\alpha\beta}(t_1',t_2'). \tag{8}$$

where $F_{\alpha\beta}$ is the term in the second brace of Eq.7.

Example We redo the same calculation of $\mathbb{C}^3$ with one box and observe some difference, perhaps due to using Eq.6 as the substitute of the character. We have easily $Q=1$, and

$$\chi=1-\frac{1}{t_1t_2t_3}+\frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3}=t_1^{-1}+t_2^{-1}+t_3^{-1}-t_1^{-1}t_2^{-1}-t_1^{-1}t_3^{-1}-t_2^{-1}t_3^{-1},$$

and we have the result quoted in Eq.4.

Example Now we consider the partition such that there is one row of box along the edge. Then

$$Q_\alpha=\frac{1}{1-t_3},\quad Q_\beta=Q_\alpha(t_3^{-1})$$

and so

$$\begin{aligned}\chi_\alpha &= \frac{1}{1-t_3}-\frac{1}{1-t_3^{-1}}\frac{1}{t_1t_2t_3}+\frac{1}{(1-t_3)(1-t_3^{-1})}\frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3},\\ &= \frac{1}{1-t_3}+\frac{1}{(1-t_3)t_1t_2}-\frac{1}{(1-t_3)}\frac{(1-t_1)(1-t_2)}{t_1t_2},\\ &= \frac{1}{1-t_3}(\frac{1}{t_1}+\frac{1}{t_2}).\end{aligned}$$

For the edge

$$\delta(1-t_3)\Big(-1-\frac{1}{t_1t_2}+\frac{(1-t_1)(1-t_2)}{t_1t_2}\Big)=\delta(1-t_3)\Big(-\frac{1}{t_1}-\frac{1}{t_2}\Big).$$

Combining the vertices and the edge, we get contribution only from the β vertex

$$\begin{aligned}\chi &= \chi_\alpha(t') + \frac{t_3^{-1}}{1-t_3^{-1}}\Big(-\frac{1}{t_1}-\frac{1}{t_2}\Big) = \frac{1}{1-t_3^{-1}}\Big(\frac{1}{t_1 t_3^a}+\frac{1}{t_2 t_3^b}\Big) - \frac{t_3^{-1}}{1-t_3^{-1}}\Big(\frac{1}{t_1}+\frac{1}{t_2}\Big) \\ &= \frac{1}{1-t_3}\Big(\frac{1}{t_1}(1-t_3^{-a+1}) + \frac{1}{t_2}(1-t_3^{-b+1})\Big).\end{aligned}$$

Now if $a=b=1$, i.e. we have $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$ then $\chi=0$. Now if $a=k, b=2-k, k\geq 2$, then

$$\chi = -\frac{1}{t_1}(t_3^{-1}+\cdots+t_3^{1-k}) + \frac{1}{t_2}(1+\cdots+t_3^{k-2}),$$

from which we get the Euler character

$$e = \prod_{i=1}^{k-1}(-\omega_1-\omega_3 j)^{-1}\prod_{i=0}^{k-2}(-\omega_2+\omega_3 j) = \frac{(-\omega_2)(-\omega_2+\omega_3)\cdots(-\omega_2+(k-2)\omega_3)}{(-\omega_1-\omega_3)\cdots(-\omega_1-(k-1)\omega_3)}.$$

If we restrict to the sub-torus $\sum_i \omega_i = 0$ as above, then $e=(-1)^{k-1}$, agreeing also with the case $k=1$.

Example Let there now be a row of p boxes along the edge, and let $Q(t_1,t_2)$ be the character of these boxes along the edge. As in Eq.8,

$$\begin{aligned}\chi &= -\frac{t_3^{-1}}{1-t_3^{-1}}F_{\alpha\beta}(t_1,t_2) - \frac{1}{1-t_3^{-1}}F_{\alpha\beta}(t_1',t_2'), \\ F(t_1,t_2) &= -Q(t_1,t_2) - \frac{Q(t_1^{-1},t_2^{-1})}{t_1t_2} + Q(t_1,t_2)Q(t_1^{-1},t_2^{-1})\frac{(1-t_1)(1-t_2)}{t_1t_2}.\end{aligned}$$

If $Q(t_1,t_2) = 1+t_1$,

$$\begin{aligned}F(t_1,t_2) &= -\frac{t_1}{t_2} - \frac{1}{t_1} - \frac{1}{t_2} - \frac{1}{t_1^2} \\ &= \frac{1}{1-t_3}\Big(\frac{t_1}{t_2}(1-t_3^{a-b+1}) + \frac{1}{t_1}(1-t_3^{-a+1}) + \frac{1}{t_2}(1-t_3^{-b+1}) + \frac{1}{t_1^2}(1-t_3^{-2a+1})\Big),\end{aligned}$$

letting again $a=1=b$

$$\chi = \frac{t_1}{t_2} - \frac{1}{t_1^2 t_3}, \qquad e = \frac{(\omega_1-\omega_2)}{-(2\omega_1+\omega_3)} = -1.$$

Also $a=k,\ b=2-k,\ k\geq 2$

$$\begin{aligned}&\chi = \frac{t_1}{t_2}(1+\cdots t_3^{2k-2}) - \frac{1}{t_1}(t_3^{-1}+\cdots+t_3^{-k+1}) + \frac{1}{t_2}(1+\cdots+t_3^{k-2}) - \frac{1}{t_1^2}(t_3^{-1}+\cdots t_3^{-2k+1}) \\ &e = (-1)^{(2k-1)+(k-1)} = (-1)^k.\end{aligned}$$

The general formula is

$$e = (-1)^{\chi(Y)+\sum_{i\in E}\sum a_i|\lambda_i|},$$

where the sum is over the edges and $|\lambda_i|$ is the size of the partition along the edge. Note it does not matter if we choose a_i or b_i since they sum to 2. We have

$$\chi(Y) = \sum|\pi_\alpha| + \sum_{\alpha,\beta} g_{\alpha,\beta}, \quad g_{\alpha,\beta} = \sum_{(i,j)\in\lambda_{\alpha,\beta}}\big(m_{\alpha\beta}i + m'_{\alpha\beta}j + 1\big),$$

where $m_{\alpha,\beta}, m'_{\alpha,\beta}$ denote the local geometry $\mathcal{O}(m_{\alpha,\beta}) \oplus \mathcal{O}(m'_{\alpha,\beta}) \to \mathbb{P}^1$ associated to the edge, and the convention is that i, j starts from 0 at the origin. This is derived in Eq.9.

So for the first case above we have $\chi(Y) = 1$ and $e = (-1)^{1+k}$ and for the second $\chi(Y) = k$ and $e = (-1)^{k+k\cdot 2}$.

We compute now the $\chi(Y)$

$$\sum Q_\alpha + \delta(1-t_3)\sum Q_{\alpha\beta}.$$

we combine the edges with the vertices as before

$$\delta(1-t_3)Q_{\alpha\beta}(t) \;\Rightarrow\; \Big(\frac{1}{1-t_3} + \frac{t_3^{-1}}{1-t_3^{-1}}\Big)Q_{\alpha\beta}(t),$$

Let $\lambda_{\alpha\beta}$ be the asymptotic 2D partition of π_α in the β direction. For the vertices, we set $t = 1$ to get the dimension $|\pi_\alpha|$, where we subtract the asymptotic legs as above.

From the edges we have leftover

$$\frac{t_3^{-1}}{1-t_3^{-1}}Q_{\alpha\beta}(t) - \frac{1}{1-t_3^{-1}}Q_{\alpha\beta}(t').$$

For each monomial $t_1^i t_2^j$ in λ

$$\frac{t_1^i t_2^j}{1-t_3}\Big(t_3^{ai+bj+1} - 1\Big) \overset{t=1}{\Rightarrow} -(ai+bj+1).$$

Remembering the minus sign from the Čech cover we get

$$\chi(Y) = \sum_\alpha |\pi_\alpha| + \sum_{\alpha,\beta}(ai+bj+1). \tag{9}$$

References